COLOURED GRAPHICAL MODELS AND THEIR SYMMETRIES

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Coloured graphical models are Gaussian statistical models determined by an undirected coloured graph. These models can be described by linear spaces of symmetric matrices. We outline a relationship between the symmetries of the graph and the linear forms that vanish on the reciprocal variety of the model. In the special case of the uniform coloured n-cycle, this gives a complete description of all such linear forms.

1. Introduction

Coloured graphical models are a class of Gaussian statistical models first introduced in [2] to model situations where some of the covariates have approximately equal empirical concentrations. These equalities are represented by a coloured graph. The set of concentration matrices associated to a coloured graphical model is a linear subspace of the set of real symmetric $n \times n$ matrices.

Following the approach of algebraic statistics we view maximum likelihood estimation as a problem in algebraic geometry, see for instance [4, Ch. 7]. In our case, we consider the complex linear space of symmetric matrices $\mathcal{L} \subseteq \mathbb{S}^n$ defined by a given coloured graphical model and its *reciprocal variety* $\mathcal{L}^{-1}$, which is defined by the matrix inverses of the points in $\mathcal{L}$. In particular, algebraic statisticians are interested in a collection of algebraic properties of $\mathcal{L}$, such as its degree $\deg(\mathcal{L})$, its ML degree $\mathrm{mld}(\mathcal{L})$, its reciprocal ML degree $\mathrm{rmld}(\mathcal{L})$, and the ideal $I(\mathcal{L}^{-1})$ defining the reciprocal variety. For the definitions of these properties, see [1].

The authors of [3] compute some of these algebraic properties for coloured graphical models whose underlying graph is the four-cycle. Inspired by their work, we are interested in understanding these properties for models given by coloured n-cycles. These graphs often exhibit symmetries that provide algebraic information. In this article, we explore the notion of symmetry of a coloured graph and study its connection with the linear part of the ideal $I(\mathcal{L}^{-1})$.

Our main result shows that in the particularly symmetric case of the *uniform coloured n-cycle*, the graph symmetries allow us to describe the linear part of $I(\mathcal{L}^{-1})$ completely.

Notation. Let $G = (V,E)$ be a graph. A *colouring* of G is a partition of the vertex set V together with a partition of the edge set E. A *colour* is an equivalence class with respect to one of these partitions.

Let G be a coloured graph with n vertices and d colours $\gamma_1,\dots,\gamma_d$. For $k = 1,\dots,d$ we define the matrix A_k associated to the colour γ_k as follows. If γ_k is a vertex colour then $(A_k)_{ii} = 1$ if vertex i has the colour γ_k, while the other entries are all zero. If γ_k is an edge colour then $(A_k)_{ij} = 1$ if edge (i, j) exists and has the colour γ_k, and all the other entries are zero. The *adjacency matrix* A of G is the matrix

$$\sum_{k=1}^{d} \lambda_k A_k \in \mathbb{Z}[\lambda_1,\dots,\lambda_d]^{n\times n}.$$

Let G be a coloured graph with n vertices and d colours. The associated linear space of symmetric matrices $\mathcal{L}$ is the subspace

$$\left\{ \sum_{k=1}^{d} \lambda_k A_k : \lambda_1,\dots,\lambda_d \in \mathbb{C} \right\}$$

of the space of complex symmetric $n\times n$ matrices $\mathbb{S}^n$. The *reciprocal variety* $\mathcal{L}^{-1}$ is the Zariski-closure of the set

$$\{A^{-1} : A \in \mathcal{L} \text{ invertible}\} \subseteq \mathbb{S}^n.$$

Its ideal is denoted by $I(\mathcal{L}^{-1})$. A generic element of $\mathcal{L}^{-1}$ has the form $\operatorname{adj}(A)$ for $A \in \mathcal{L}$ invertible, where $\operatorname{adj}(A)$ is the classical adjoint of A.

2. Linear forms from symmetries

We start by defining graph symmetries, following the approach in [2, Sec. 5].

Definition 2.1. Let A be the adjacency matrix of a coloured graph G. A *symmetry* of G is an $n\times n$ permutation matrix B such that $BAB^{-1} = A$.

Next, we show that such a symmetry gives a set of linear forms that vanish on the reciprocal variety $\mathcal{L}^{-1}$.

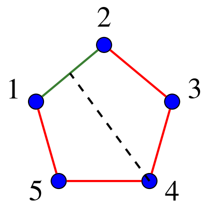
Proposition 2.2. *Let B be a symmetry of G and X a generic element of $\mathbb{S}^n$. The binomial linear forms defined by all distinct entries of the matrix $BXB^{-1} - X$ belong to $I(\mathcal{L}^{-1})$.*

Proof. By our assumption, we have $BA = AB$. Hence,

$$\operatorname{adj}(A)B\det(A) = \operatorname{adj}(A)BA\operatorname{adj}(A) = \det(A)B\operatorname{adj}(A).$$

Since $\det A \neq 0$, we have $B\operatorname{adj}(A)B^{-1} - \operatorname{adj}(A) = 0$. Thus, a generic element $X = \operatorname{adj} A$ of $\mathcal{L}^{-1}$ must satisfy the equation $BXB^{-1} - X = 0$, whose entries are binomial linear forms. □

Example 2.3. Let G be the following coloured 5-cycle:



The only nontrivial symmetry of G is the reflection B_τ along the depicted axis. The matrix $B_\tau X B_\tau^{-1} - X$ equals

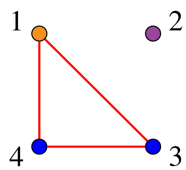
$$\begin{pmatrix} -x_{11}+x_{22} & 0 & -x_{13}+x_{25} & -x_{14}+x_{24} & -x_{15}+x_{23} \\ 0 & x_{11}-x_{22} & x_{15}-x_{23} & x_{14}-x_{24} & x_{13}-x_{25} \\ -x_{13}+x_{25} & x_{15}-x_{23} & -x_{33}+x_{55} & -x_{34}+x_{45} & 0 \\ -x_{14}+x_{24} & x_{14}-x_{24} & -x_{34}+x_{45} & 0 & x_{34}-x_{45} \\ -x_{15}+x_{23} & x_{13}-x_{25} & 0 & x_{34}-x_{45} & x_{33}-x_{55} \end{pmatrix}.$$

By Proposition 2.2, the following binomial linear forms are in $I(\mathcal{L}^{-1})$:

$$x_{11}-x_{22}, \quad x_{33}-x_{55}, \quad x_{13}-x_{25}, \quad x_{14}-x_{24}, \quad x_{15}-x_{23}, \quad x_{34}-x_{25}.$$

These do not generate the whole linear part of $I(\mathcal{L}^{-1})$. For instance, the linear form $x_{14}+x_{44}-x_{35}-x_{55}$ is contained in the ideal but is not a linear combination of the above.

Example 2.4. Let G be the following disconnected graph:



The only nontrivial symmetry of G is the permutation B_ρ of vertices 3 and 4. The matrix $B_\rho X B_\rho^{-1} - X$ gives all the binomial linear forms in the following list of generators of the linear part of $I(\mathcal{L}^{-1})$:

$$x_{33}-x_{44}, \quad x_{13}-x_{14}, \quad x_{12}, \quad x_{23}, \quad x_{24}.$$

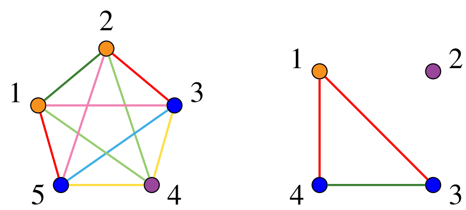
In the following proposition we also provide an explanation for the 3 other generators.

Proposition 2.5. *If i is an isolated vertex of the graph G then for all $j \neq i$ we have $x_{ij} \in I(\mathcal{L}^{-1})$.*

Proof. If i is an isolated vertex then for all matrices $A \in \mathcal{L}$, the i-th row and i-th column will have only zero entries except for the entry a_{ii}. It follows that the submatrix A_{ij} given by deleting the i-th row and j-th column will contain a zero column, so $\det(A_{ij}) = 0$ and therefore $x_{ij} = 0$ for all $X \in \mathcal{L}^{-1}$. □

Let I' be the ideal generated by all linear forms found by applying Proposition 2.2 to all symmetries of G and Proposition 2.5 to all its isolated vertices. We define a further linear subspace $\mathcal{L}'$ as the vanishing set $V(I')$. Like $\mathcal{L}$, this linear subspace is associated to a coloured graph G', whose colouring is induced by the linear forms in I'.

Example 2.6. The linear forms found in Examples 2.3 and 2.4 give rise to the following graphs G_1' and G_2'.



By Proposition 2.2, every symmetry of a coloured graph G is also a symmetry of G'. We also see that both $\mathcal{L}$ and $\mathcal{L}^{-1}$ are contained in $\mathcal{L}'$. The next proposition demonstrates how $\mathcal{L}'$ can be used as a lower-dimensional ambient space to compute the ML degree of $\mathcal{L}$.

Proposition 2.7. *Let S' be a general matrix of $\mathcal{L}'$ and let $(\mathcal{L}^\perp)' = \mathcal{L}^\perp \cap \mathcal{L}'$, where $\mathcal{L}^\perp \subseteq \mathbb{S}^n$ is the orthogonal complement to $\mathcal{L}$ with respect to the trace inner product. Let $(\mathcal{L}^\perp)' + S'$ denote the affine space $\{X' + S' \mid X' \in (\mathcal{L}^\perp)'\}$. The ML degree of $\mathcal{L}$ is the cardinality of the set*

$$\mathcal{L}^{-1} \cap ((\mathcal{L}^\perp)' + S').$$

Proof. By definition, the ML degree of $\mathcal{L}$ is the cardinality of $\mathcal{L}^{-1} \cap (\mathcal{L}^\perp + S)$, where S is a general matrix of $\mathbb{S}^n$ and $\mathcal{L}^\perp \subseteq \mathbb{S}^n$ is the orthogonal complement of $\mathcal{L}$ in $\mathbb{S}^n$. Since $\mathcal{L} \subseteq \mathcal{L}'$, we have $\operatorname{span}(\mathcal{L}', \mathcal{L}^\perp) = \mathbb{S}^n$. We write all matrices $M \in \mathbb{S}^n$ as $M = M_1 + M_2 + M_3$, where the term M_1 is the part of M contained in $(\mathcal{L}^\perp)'$, the term M_2 is the remaining part contained in $\mathcal{L}^\perp$, and M_3 is the remaining part contained in $\mathcal{L}'$. Let $S' = S_1 + S_3$. Since S is general in $\mathbb{S}^n$, the matrix S' is general in $\mathcal{L}'$, so it remains to show that

$$(\mathcal{L}^\perp + S) \cap \mathcal{L}' = ((\mathcal{L}^\perp)' + S').$$

The reverse inclusion is clear. For the forward inclusion, Let $\Sigma \in (\mathcal{L}^{\perp}+S)\cap\mathcal{L}'$ and write $\Sigma = X+S$ with $X \in \mathcal{L}^{\perp}$. Write $\Sigma = X_1+X_2+S'+S_2$ using the above decomposition for X and S. Since $\Sigma \in \mathcal{L}'$, we have $X_2+S_2 \in \mathcal{L}'$. By construction, $X_2+S_2 = 0$, so $\Sigma = X_1+S' \in ((\mathcal{L}^{\perp})'+S')$, as required. □

3. The uniform coloured n-cycle

Proposition 2.2 describes how to derive linear forms of $I(\mathcal{L}^{-1})$ from the symmetries of G. In the previous section, we saw that these need not generate the whole linear part of $I(\mathcal{L}^{-1})$. In this section, we present a class of examples where they do.

The *uniform coloured n-cycle* is the coloured n-cycle G whose colouring has only one edge colour γ_1 and one vertex colour γ_2. For a picture when $n = 5$, see the first entry of Table 2. The associated linear space of symmetric matrices $\mathcal{L}$ is spanned by the matrices A_1 and A_2 where

$$A_1 = \begin{pmatrix} 0 & 1 & & 1 \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & 0 \end{pmatrix}$$

and $A_2 = \mathrm{id}$. It is in particular a *pencil of quadrics* as studied in [1]. A stratification of all regular pencils of quadrics is given in [1]. This stratification is indexed by the *Segre symbol* σ, which also determines many algebraic properties of $\mathcal{L}$.

Definition 3.1 ([1])**.** Let $\mathcal{L}$ be a pencil of quadrics spanned by the matrices A_1 and A_2, such that A_2 is an invertible matrix. The *Segre symbol* of $\mathcal{L}$ is the unordered list

$$\sigma = [\sigma_1 \dots \sigma_r]$$

specified by the Jordan canonical form of $A_1A_2^{-1}$ in the following way. Let $\alpha_1,\dots,\alpha_r$ be the distinct eigenvalues of $A_1A_2^{-1}$. Each σ_i is itself a tuple of natural numbers associated to the eigenvalue α_i. The entries of σ_i are precisely the sizes of the Jordan blocks associated to α_i.

The Segre symbol of the linear space $\mathcal{L}$ associated to the uniform coloured n-cycle is therefore a description of the Jordan canonical form of the matrix A_1 above. Since A_1 is a symmetric matrix, it is diagonalisable and therefore the number of Jordan blocks will be n and all σ_i will be 1's or bracketed lists of 1's.

In this section, we will use the following symmetry of G, which represents an anti-clockwise rotation of the vertices:

$$B = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}.$$

Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$. The matrices A_1 and B satisfy the following equations, where all indices are taken modulo n:

$$\begin{aligned} A_1(e_i) &= e_{i-1} + e_{i+1} \\ B(e_i) &= e_{i+1}. \end{aligned}$$

Lemma 3.2. *The matrices $B - \mathrm{id}$ and $(B - \mathrm{id})^2$ have rank $n-1$. The matrices $B + \mathrm{id}$ and $(B + \mathrm{id})^2$ have rank $n-1$ if n is even, rank n otherwise.*

Proof. We start with the statement about the matrices $B - \mathrm{id}$ and $B + \mathrm{id}$:

$$\begin{pmatrix} -1 & & & 1 \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & & 1 \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix}.$$

Adding the sum of rows 2 to n to the first row of $B - \mathrm{id}$ gives a matrix of rank $n-1$. Adding the alternating sum, starting with a minus, of rows 2 to n to the first row of $B + \mathrm{id}$ gives a matrix of rank $n-1$ if n is even, of rank n otherwise.

Next, we prove the statement about $(B - \mathrm{id})^2$. Let $v \in \ker(B - \mathrm{id})^2$. The vectors $(1, \dots, 1)^T$ and $w := (B - \mathrm{id})v$ are both contained in $\ker(B - \mathrm{id})$, so we have $w = \mu(1, \dots, 1)^T$ for some scalar μ. Hence we have $Bv - v - \mu(1, \dots, 1)^T = 0$. This is a linear system of equations in the entries of v and in μ. It is represented by the matrix on the left in the following picture, which after the same row operations done above for $B - \mathrm{id}$ reduces to the matrix on the right:

$$\begin{pmatrix} -1 & \cdots & \cdots & 1 & -1 \\ 1 & -1 & & & \vdots \\ & \ddots & \ddots & & \vdots \\ & & 1 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & \cdots & \cdots & 0 & -n \\ 1 & -1 & & & -1 \\ & \ddots & \ddots & & \vdots \\ & & 1 & -1 & -1 \end{pmatrix}.$$

From this we see that μ must be zero, so $v \in \ker(B - \mathrm{id})$. Hence, we have $\ker(B - \mathrm{id})^2 = \ker(B - \mathrm{id})$, which implies that $(B - \mathrm{id})^2$ has rank $n-1$.

The statement about $(B+\mathrm{id})^2$ when n is odd follows from the fact that $B+\mathrm{id}$ has rank n in that case. If n is even, it follows by essentially the same argument as above, with $w := (B+\mathrm{id})v$ now having to satisfy $w = \mu(1,-1,\dots 1,-1)^T$. After the same row operations as for $B+\mathrm{id}$ above, the corresponding linear system $Bv+v-\mu(1,-1,\dots 1,-1)^T$ reduces to one whose matrix has $(0,\dots,0,-n)$ as its first row. This shows that $\mu = 0$ and $\ker(B+\mathrm{id})^2 = \ker(B+\mathrm{id})$. □

Lemma 3.3. *The matrix A_1 always has the eigenvalue 2 with geometric multiplicity one. It has the eigenvalue -2 if and only if n is even, in which case the geometric multiplicity is one.*

Proof. We have $A_1(\sum_i e_i) = 2\sum_i e_i$ always, and $A_1(\sum_i(-1)^i e_i) = -2\sum_i(-1)^i e_i$ if n is even. Since $A_1 = B^{-1}+B$, we have $B(A_1 \pm 2\,\mathrm{id}) = (B\pm\mathrm{id})^2$. Hence we have $\ker(A_1 \pm 2\,\mathrm{id}) = \ker(B\pm\mathrm{id})^2$ and the statement follows from Lemma 3.2. □

Lemma 3.4. *Let $\alpha \neq 2,-2$ be an eigenvalue of A_1. Then the geometric multiplicity of α is two.*

Proof. Let v be an eigenvector of A_1 to the eigenvalue α. Then Bv is also an eigenvector to α, since A_1 commutes with B by Proposition 2.2. The vectors v and Bv are linearly independent because otherwise v would be an eigenvector of B, which does not have other real eigenvalues than 1 and possibly -1. The first case would imply $\alpha = 2$ and the second $\alpha = -2$. So the geometric multiplicity of α is at least two. It is not greater than two because the matrix $A_1 - \alpha\,\mathrm{id}$,

$$\begin{pmatrix} -\alpha & 1 & & & 1 \\ 1 & -\alpha & \ddots & & \\ & 1 & \ddots & 1 & \\ & & \ddots & -\alpha & 1 \\ 1 & & & 1 & -\alpha \end{pmatrix},$$

has $n-2$ linearly independent columns: the middle ones. □

Proposition 3.5. *Let $\mathcal{L}$ be the pencil of quadrics associated to the uniform coloured n-cycle G, σ its Segre symbol, and r the length of σ. Then*

$$\text{For } n \text{ even:} \quad r = \frac{n+2}{2}, \quad \sigma = [(1,1),\dots,(1,1),1,1]\,;$$
$$\text{For } n \text{ odd:} \quad r = \frac{n+1}{2}, \quad \sigma = [(1,1),\dots,(1,1),1]\,.$$

Proof. Lemma 3.3 accounts for the lone 1's and by Lemma 3.4 the remaining entries are pairs of bracketed 1's. Since the matrix A_1 has n Jordan blocks, this also determines the length r. □

	n even	n odd
$\mathrm{mld}(\mathcal{L})$	$\frac{n}{2}$	$\frac{n-1}{2}$
$\mathrm{rmld}(\mathcal{L})$	$n-1$	$n-2$
$\deg(\mathcal{L}^{-1})$	$\frac{n}{2}$	$\frac{n-1}{2}$
no. linear forms	$\frac{n^2-2}{2}$	$\frac{n^2-1}{2}$
no. quadratic forms	$\frac{n(n-2)}{8}$	$\frac{(n-1)(n-3)}{8}$

Table 1: Algebraic properties of the linear space of symmetric matrices $\mathcal{L}$ associated to the coloured n-cycle. The first two rows are the ML and reciprocal ML degree of $\mathcal{L}$ as defined by [1]. The middle row is the degree of $\mathcal{L}^{-1}$. The last two rows give the number of linear and quadratic forms in a set of minimal generators of the ideal $I(\mathcal{L}^{-1})$.

Following [1], the Segre symbols computed in Proposition 3.5 determine algebraic properties of $\mathcal{L}$. We summarize these in Table 1. Since $\mathcal{L}$ is a pencil, the ideal $I(\mathcal{L}^{-1})$ is generated by linear and quadratic forms. Knowing the number of linear forms in a minimal generating set, we can now prove our main result.

Theorem 3.6. *The linear part of $I(\mathcal{L}^{-1})$ is generated by the linear forms*

$$x_{i,i+d} - x_{1,1+d}, \tag{1}$$

for $i = 2, \dots, n$ and $d = 0, \dots, \lfloor n/2 \rfloor$, where all indices are taken modulo n and x_{ji} for $j > i$ is taken to mean x_{ij}.

Proof. Proposition 2.2 says that the linear forms in the entries of

$$BXB^{-1} - X = \begin{pmatrix} x_{nn} & x_{1n} & \cdots & x_{n-1,n} \\ x_{1n} & x_{11} & \cdots & x_{1,n-1} \\ \vdots & \vdots & & \vdots \\ x_{n-1,n} & x_{1,n-1} & \cdots & x_{n-1,n-1} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{pmatrix}$$

are all contained in $I(\mathcal{L}^{-1})$. A generating set for the linear forms thus described is given by the linear forms in (1). When n is odd, there are $n-1$ distinct forms for each $d = 0, \dots, (n-1)/2$. Since the number of these linear forms matches the expression given in Table 1, we have a complete generating set. For n even and $d < n/2$ there are $n-1$ distinct linear forms. For $d = n/2$, there are only $(n-2)/2$ distinct nontrivial linear forms. Namely, for $i = n/2$ the given linear form is trivial, and the linear forms for $i < n/2$ are the same as the ones for $i > n/2$. Taken together, we have $(n-1)(n/2)+(n-2)/2$ distinct linear forms, which again matches the expression in Table 1. □

Example 3.7. The linear space associated to the uniform coloured 6-cycle is

$$\mathcal{L} = \left\{ \begin{pmatrix} \lambda_2 & \lambda_1 & 0 & 0 & 0 & \lambda_1 \\ \lambda_1 & \lambda_2 & \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & \lambda_2 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & \lambda_2 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 & \lambda_2 & \lambda_1 \\ \lambda_1 & 0 & 0 & 0 & \lambda_1 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

The Jordan canonical form of A_1 is $\mathrm{diag}(1,1,-1,-1,2,-1)$ and the Segre symbol is $\sigma = [(1,1),(1,1),1,1]$. According to Theorem 3.6, the following 17 linear forms generate the linear part of $I(\mathcal{L}^{-1})$:

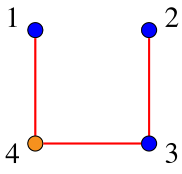
$$\begin{array}{llll} x_{22}-x_{11} & x_{23}-x_{12} & x_{24}-x_{13} & x_{25}-x_{14} \\ x_{33}-x_{11} & x_{34}-x_{12} & x_{35}-x_{13} & x_{36}-x_{14} \\ x_{44}-x_{11} & x_{45}-x_{12} & x_{46}-x_{13} & \\ x_{55}-x_{11} & x_{56}-x_{12} & x_{15}-x_{13} & \\ x_{66}-x_{11} & x_{16}-x_{12} & x_{26}-x_{13} & \end{array}$$

The graph G' associated to the symmetries of G can be found in Table 4.

4. Open Problems

By Proposition 2.2, the symmetries of a coloured graph explain some binomial linear forms in the ideal $I(\mathcal{L}^{-1})$. The following example shows that in general, not all binomial linear forms in $I(\mathcal{L}^{-1})$ come from such symmetries.

Example 4.1. The graph



has no nontrivial symmetries, but all elements $X \in \mathcal{L}^{-1}$ satisfy $x_{13} = x_{24}$.

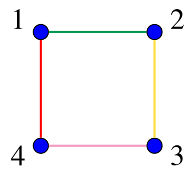
However, after computing the linear part of $I(\mathcal{L}^{-1})$ for all colourings on the 3,4, and 5-cycle, we arrived at the following conjecture.

Conjecture 4.2. Let G be a coloured n-cycle. All *binomial* linear forms in $I(\mathcal{L}^{-1})$ can be found using Proposition 2.2.

Tables 2 and 3 give a complete list of all coloured 5-cycles with only one vertex colour. Just as in Example 2.3, some linear forms cannot be described by Propositions 2.2 and 2.5. We have highlighted these.

Question 4.3. Let G be a coloured n-cycle. Is there a graphical explanation for all generators of the linear part of $I(\mathcal{L}^{-1})$?

Example 4.4. Consider the following coloured 4-cycle G:



The linear part of $I(\mathcal{L}^{-1})$ is generated by the linear form $x_{11}+x_{33}-x_{22}-x_{44}$. A graphical explanation for this linear form could be that the set of edges adjacent to vertices 1 and 3 coincides with the set of edges adjacent to vertices 2 and 4.

In this article we have focused on linear forms. We do not know whether the quadratic forms of a minimal generating set of $I(\mathcal{L}^{-1})$ can be deduced graphically from G, even in the special case of the uniform coloured n-cycle. In Table 4 we give these quadratic forms when G is the uniform coloured n-cycle for $n = 3,\dots,8$, along with the graph G' associated to the symmetries of G.

Acknowledgements. We thank Shiyue Li, Aida Maraj and Bernd Sturmfels for helpful conversations in the early stages of this project.

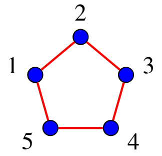
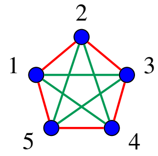
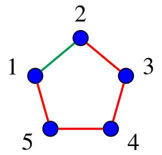
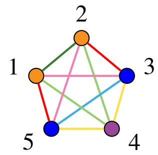
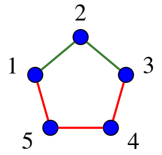
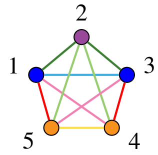
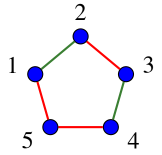
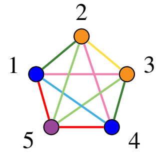
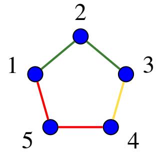
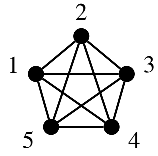
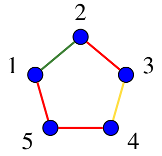
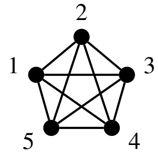
G	Linear Forms	G'
	$x_{11}-x_{55}$ $x_{12}-x_{45}$ $x_{13}-x_{35}$ $x_{22}-x_{55}$ $x_{23}-x_{45}$ $x_{14}-x_{35}$ $x_{33}-x_{55}$ $x_{34}-x_{45}$ $x_{24}-x_{35}$ $x_{44}-x_{55}$ $x_{15}-x_{45}$ $x_{25}-x_{35}$	
	$x_{11}-x_{22}$ $x_{15}-x_{23}$ $x_{33}-x_{55}$ $x_{34}-x_{45}$ $x_{13}-x_{25}$ $x_{14}-x_{24}$ $x_{24}+x_{44}-x_{35}-x_{55}$	
	$x_{11}-x_{33}$ $x_{14}-x_{35}$ $x_{44}-x_{55}$ $x_{15}-x_{34}$ $x_{12}-x_{23}$ $x_{24}-x_{25}$ $x_{13}+x_{34}+x_{55}-x_{33}-x_{35}-x_{45}$	
	$x_{11}-x_{44}$ $x_{13}-x_{24}$ $x_{22}-x_{33}$ $x_{15}-x_{45}$ $x_{12}-x_{34}$ $x_{25}-x_{35}$	
	None.	
	None.	

Table 2: Linear minimal generators of $I(\mathcal{L}^{-1})$ for coloured five-cycles. The left column of the table shows the given coloured five-cycle G with all vertices having the same colour. The linear generators of the corresponding reciprocal varieties are shown in the middle column. The generators not explained by Proposition 2.2 are highlighted. The right column shows the graph G' associated to the symmetries of G. If I' is the zero ideal, we give G' as the uncoloured complete graph on 5 vertices.

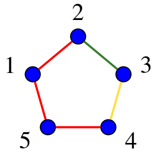
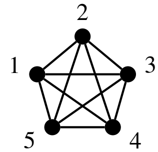
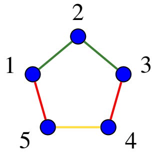
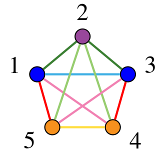
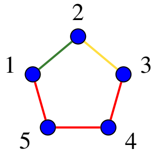
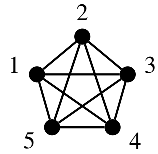
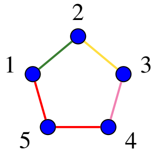
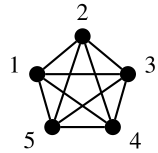
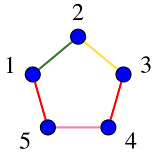
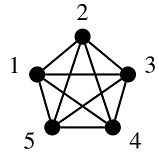
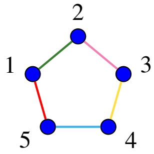
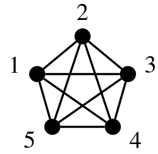
G	Linear Forms	G'
	None.	
	$x_{11}-x_{33}$ $x_{14}-x_{35}$ $x_{44}-x_{55}$ $x_{15}-x_{34}$ $x_{12}-x_{23}$ $x_{24}-x_{25}$	
	$x_{14}+x_{44}-x_{35}-x_{55}$	
	None.	
	None.	
	None.	

Table 3: Continuation of Table 2.

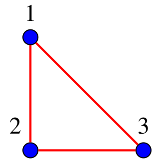
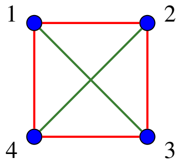
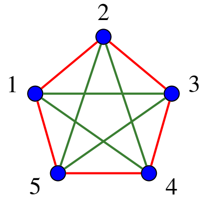
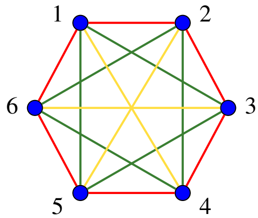
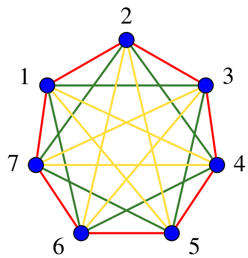
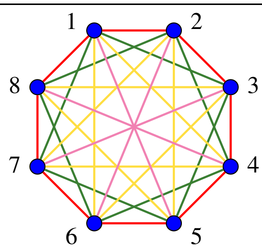
n	Quadratic Forms	Graph G'
3	None.	
4	$x_{13}^2 - 2x_{12}^2 + x_{13}x_{11}$	
5	$x_{13}^2 - x_{13}x_{12} - x_{12}^2 + x_{13}x_{11}$	
6	$2x_{13}^2 - x_{12}x_{14} - x_{14}^2$ $2x_{12}x_{13} - x_{11}x_{14} - x_{13}x_{14}$ $2x_{12}^2 - 2x_{11}x_{13} + x_{12}x_{14} - x_{14}^2$	
7	$x_{13}^2 - x_{12}x_{14} + x_{13}x_{14} - x_{14}^2$ $x_{12}x_{13} - x_{11}x_{14} + x_{12}x_{14} - x_{13}x_{14}$ $x_{12}^2 - x_{11}x_{13} + x_{13}x_{14} - x_{14}^2$	
8	$2x_{14}^2 - x_{13}x_{15} - x_{15}^2$ $2x_{13}x_{14} - x_{12}x_{15} - x_{14}x_{15}$ $2x_{12}x_{14} - x_{11}x_{15} - x_{13}x_{15}$ $2x_{13}^2 - x_{11}x_{15} - x_{15}^2$ $2x_{12}x_{13} - 2x_{11}x_{14} + x_{12}x_{15} - x_{14}x_{15}$ $2x_{12}^2 - 2x_{11}x_{13} + x_{13}x_{15} - x_{15}^2$	

Table 4: Quadratic forms in a minimal generating set of $I(\mathcal{L}^{-1})$ for the first eight uniform coloured n-cycles. The right column displays the graph G' representing all linear forms in the minimal generating set (by Theorem 3.6).

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