

# VERLINDE BUNDLES OF FAMILIES OF HYPERSURFACES AND THEIR JUMPING LINES

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## Abstract

Verlinde bundles are vector bundles  $V_k$  arising as the direct image  $\pi_*(\mathcal{L}^{\otimes k})$  of polarizations of a proper family of schemes  $\pi: \mathfrak{X} \rightarrow S$ . We study the splitting behavior of Verlinde bundles in the case where  $\pi$  is the universal family  $\mathfrak{X} \rightarrow |\mathcal{O}(d)|$  of hypersurfaces of degree  $d$  in  $\mathbb{P}^n$  and calculate the cohomology class of the locus of jumping lines of the Verlinde bundles  $V_{d+1}$  in the cases  $n = 2, 3$ .

## 1 Introduction

Let  $\pi: \mathfrak{X} \rightarrow S$  be a proper family of schemes with a polarization  $\mathcal{L}$ . For  $k \geq 1$ , if the sheaf  $\pi_*(\mathcal{L}^{\otimes k})$  is locally free, we call it the  $k$ -th *Verlinde bundle* of the family  $\pi$ .

For example ([Iye13]), let  $C \rightarrow T$  be a smooth projective family of curves of fixed genus. Consider the relative moduli space  $\pi: \mathrm{SU}(r) \rightarrow T$  of semistable vector bundles of rank  $r$  and trivial determinant. This family is equipped with a polarization  $\Theta$ , the determinant bundle. The Verlinde bundles  $\pi_*(\Theta^k)$  of this family are projectively flat ([Hit90],[ADPW91]), and their rank is given by the Verlinde formula.

In this article, we study the example of the universal family  $\pi: \mathfrak{X} \rightarrow |\mathcal{O}_{\mathbb{P}^n}(d)|$  of hypersurfaces of degree  $d$  in the complex projective space  $\mathbb{P}^n$ , with  $n > 1$ . This family comes equipped with the polarization  $\mathcal{L}$  given by the pullback of  $\mathcal{O}(1)$  along the projection map  $\mathfrak{X} \rightarrow \mathbb{P}^n$ . For  $k \geq 1$ , the sheaf  $\pi_*\mathcal{L}^{\otimes k}$  is locally free, as can be seen by considering the structure sequence of an arbitrary hypersurface of degree  $d$  in  $\mathbb{P}^n$ . For  $k \geq 1$ , we denote the  $k$ -th Verlinde bundle of the family  $\pi$  by  $V_k$ .

To better understand  $V_k$  we study its splitting type when restricted to lines in  $|\mathcal{O}(d)|$ .

Let  $T \subseteq |\mathcal{O}(d)|$  be a line. On  $T = \mathbb{P}^1$ , we define the vector bundle  $V_{k,T} := V_k|_T$ . The *splitting type* of  $V_{k,T}$  is the unique non-increasing tuple  $(b_1, \dots, b_{r^{(k)}})$  of size  $r^{(k)} := \mathrm{rk} V_k$  such that  $V_{k,T} \simeq \bigoplus_i \mathcal{O}(b_i)$ .

The sequence (2.1) puts constraints on the  $b_i$ : they are all non-negative and they sum up to  $d^{(k)} := \deg(V_k)$ . The set of such tuples  $(b_i)$  can be ordered by defining the expression  $(b'_i) \geq (b_i)$  to mean

$$\sum_{i=1}^s b'_i \geq \sum_{i=1}^s b_i \text{ for all } s = 1, \dots, r.$$

With this definition, smaller types are more general: the vector bundle  $\mathcal{O}(b_i)$  on  $\mathbb{P}^1$  specializes to  $\mathcal{O}(b'_i)$  in the sense of [Sha76] if and only if  $(b'_i) \geq (b_i)$ .

If  $d^{(k)} \leq r^{(k)}$ , then the most generic possible type has thus the form  $(1, \dots, 1, 0, \dots, 0)$ . We call this the *generic splitting type*. A computation shows that  $d^{(k)} \leq r^{(k)}$  if  $k \leq 2d$ .

We have the following result on the cohomology class of the *set of jumping lines*

$$Z := \{T \in \text{Gr}(1, |\mathcal{O}(d)|) \mid V_{d+1, T} \text{ has non-generic type}\}$$

in the Grassmannian of lines in  $|\mathcal{O}(d)|$ :

**Theorem 1.1.** *Let  $n \leq 3$ , let  $Z$  be set of jumping lines of  $V_{d+1}$ , and let  $[Z]$  be the class of  $Z$  in the Chow ring  $\text{CH}(\text{Gr}(1, |\mathcal{O}(d)|))$ . We have*

$$\dim Z = n + 1 + \binom{d - 1 + n}{n}.$$

Furthermore, let  $b$  range over the integers with the property  $0 \leq b < \frac{\dim Z}{2}$  and define  $a = \dim Z - b$ ,  $a' = a + \frac{\text{codim } Z - \dim Z}{2}$ ,  $b' = b + \frac{\text{codim } Z - \dim Z}{2}$ .

(i) *If  $\dim Z$  is odd or  $n = 2$ , we have*

$$[Z] = \sum_{a, b} \left( \binom{a+1}{n} \binom{b+1}{n} - \binom{a+2}{n} \binom{b}{n} \right) \sigma_{a', b'}. \quad (1.1)$$

(ii) *If  $\dim Z$  is even and  $n = 3$ , we have*

$$[Z] = \sum_{a, b} \left( \binom{a+1}{n} \binom{b+1}{n} - \binom{a+2}{n} \binom{b}{n} \right) \sigma_{a', b'} + \binom{\frac{\dim Z}{2} + 2}{n} \binom{\frac{\dim Z}{2}}{n} \sigma_{\frac{\dim Z}{2}, \frac{\dim Z}{2}}.$$

The computation is carried out by the method of undetermined coefficients, leading into various calculations in the Chow ring of the Grassmannian. The assumption  $n \leq 3$  is needed for a certain dimension estimation.

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## 2 Attained splitting types

There exists a short exact sequence of vector bundles on  $|\mathcal{O}(d)|$

$$0 \rightarrow \mathcal{O}(-1) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k-d)) \xrightarrow{M} \mathcal{O} \otimes H^0(\mathbb{P}^n, \mathcal{O}(k)) \rightarrow V_k \rightarrow 0, \quad (2.1)$$

as can be seen by taking the pushforward of a twist of the structure sequence of  $\mathfrak{X}$  on  $\mathbb{P}^n \times |\mathcal{O}(d)|$ . The map  $M$  is given by multiplication by the section

$$\sum_I \alpha_I \otimes x^I \in H^0(|\mathcal{O}(d)|, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(d)).$$

In particular, we have  $r^{(k)} = \binom{k+n}{n} - \binom{k+n-d}{n}$  and  $d^{(k)} = \binom{k+n-d}{n}$ .

**Lemma 2.1.** *Let  $\mathcal{E}$  be a free  $\mathcal{O}_{\mathbb{P}^1}$ -module of finite rank, and let*

$$0 \rightarrow \mathcal{E}' \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\psi} \mathcal{E}'' \rightarrow 0$$

*be a short exact sequence of  $\mathcal{O}_{\mathbb{P}^1}$ -modules. Given a splitting  $\mathcal{E}'' = \mathcal{E}_1'' \oplus \mathcal{O}$ , we may construct a splitting  $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{O}$  such that the image of  $\varphi$  is contained in  $\mathcal{E}_1$ .*

*Proof.* Define  $\mathcal{E}_1 := \ker(\text{pr}_2 \circ \psi)$ , which is a locally free sheaf on  $\mathbb{P}^1$ . By comparing determinants in the short exact sequence  $0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{O} \rightarrow 0$  we see that  $\mathcal{E}_1$  is free, hence by an  $\text{Ext}^1$  computation the sequence splits. The property  $\text{im}(\varphi) \subseteq \mathcal{E}_1$  follows from the definition.  $\square$

**Proposition 2.2.** *Let  $f_1, f_2 \in |\mathcal{O}(d)|$  span the line  $T \subseteq |\mathcal{O}(d)|$  and let  $p$  be the number of zero entries in the splitting type of  $V_{k,T}$ . Let  $U := H^0(\mathbb{P}^n, \mathcal{O}(k-d))$ . We have*

$$p = \dim H^0(\mathbb{P}^n, \mathcal{O}(k)) - \dim(f_1U + f_2U).$$

*Proof.* Let  $s$  and  $t$  denote the homogeneous coordinates of  $\mathbb{P}^1$ . The map  $M|_T$  sends a local section  $\xi \otimes \theta$  to  $s\xi \otimes f_1\theta + t\xi \otimes f_2\theta$ . In particular, the image of  $\mathcal{O}(-1) \otimes U$  is contained in  $\mathcal{O} \otimes (f_1U + f_2U)$ . It follows that  $p \geq \dim H^0(\mathbb{P}^n, \mathcal{O}(k)) - \dim(f_1U + f_2U)$ .

To prove the other inequality, consider the induced sequence

$$0 \rightarrow \mathcal{O}(-1) \otimes U \xrightarrow{M|_T} \mathcal{O} \otimes (f_1U + f_2U) \rightarrow \mathcal{E}'' \rightarrow 0$$

and assume for a contradiction that  $\mathcal{E}'' \simeq \mathcal{E}_1'' \oplus \mathcal{O}$ . By Lemma 2.1, we have a splitting  $\mathcal{O} \otimes (f_1U + f_2U) \simeq \mathcal{E}_1 \oplus \mathcal{O}$  such that  $\text{im}(M|_T) \subseteq \mathcal{E}_1$ .

Consider the map  $\widetilde{M}|_T: (\mathcal{O} \otimes U) \oplus (\mathcal{O} \otimes U) \rightarrow \mathcal{O} \otimes (f_1U + f_2U)$  defined by

$$\widetilde{M}|_T(a \otimes \theta_1, b \otimes \theta_2) = a \otimes f_1\theta_1 + b \otimes f_2\theta_2.$$

We obtain the matrix description of  $\widetilde{M}|_T$  from the matrix description of  $M|_T$  as follows. If  $M|_T$  is represented by the matrix  $A$  with coefficients  $A_{i,j} = \lambda_{i,j}s + \mu_{i,j}t$ , then  $\widetilde{M}|_T$  is represented by a block matrix

$$B = \left( \begin{array}{c|c} A' & A'' \end{array} \right)$$

with  $A'_{i,j} = \lambda_{i,j}$  and  $A''_{i,j} = \mu_{i,j}$ .

The property  $\text{im}(M|_T) \subseteq \mathcal{E}_1$  implies that after some row operations, the matrix  $A$  has a zero row. By the construction of  $\widetilde{M}|_T$ , the same row operations lead to the matrix  $B$  having a zero row, but this is a contradiction, since the map  $\widetilde{M}|_T$  is surjective.  $\square$

**Corollary 2.3.** *Let  $T \subseteq |\mathcal{O}(d)|$  be a line spanned by the polynomials  $f_1, f_2$ . Assume that  $d^{(k)} \leq r^{(k)}$ . Let  $\theta$  range over a monomial basis of  $H^0(\mathbb{P}^n, \mathcal{O}(k-d))$ . The bundle  $V_{k,T}$  has the generic splitting type if and only if  $\langle f_1\theta, f_2\theta \mid \theta \rangle$  is a linearly independent set in  $H^0(\mathbb{P}^n, \mathcal{O}(k))$ .  $\square$*

**Corollary 2.4.** *Let  $T \subseteq |\mathcal{O}(d)|$  be a line spanned by the polynomials  $f_1, f_2$ , and let  $d^{(k)} \leq r^{(k)}$ . The bundle  $V_{k,T}$  has not the generic type if and only if  $\deg(\text{gcd}(f_1, f_2)) \geq 2d - k$ . In particular, if  $d^{(k)} \leq r^{(k)}$  but  $k > 2d$  then the generic type never occurs.*

*Proof.* By Corollary 2.3, the bundle  $V_{k,T}$  has non-generic type if and only if there exist linearly independent  $g_1, g_2 \in H^0(\mathbb{P}^n, \mathcal{O}(k-d))$  such that  $g_1f_1 + g_2f_2 = 0$ . Let  $h := \text{gcd}(f_1, f_2)$  and  $d' := \deg h$ .

If  $d' \geq 2d - k$  then  $\deg(f_i/h) \leq k - d$  and we may take  $g_1, g_2$  to be multiples of  $f_1/h$  and  $f_2/h$ , respectively.

On the other hand, given such  $g_1$  and  $g_2$ , we have  $f_1 \mid g_2f_2$ , which implies  $f_1/h \mid g_2$ , hence  $d - d' \leq k - d$ .  $\square$

**Proposition 2.5.** *Let  $k = d + 1$ . No types of  $V_k$  other than  $(1, \dots, 1, 0, \dots, 0)$  and  $(2, 1, \dots, 1, 0, \dots, 0)$  occur.*

*Proof.* Assume that the type of  $V_k$  at some line  $(f_1, f_2)$  is other than the two above. Then the type has at least two more zero entries than the general type. By Proposition 2.2, we have  $\dim \langle f_1\theta, f_2\theta \mid \theta \rangle \leq 2d^{(k)} - 2$ , so we find  $g_1, g_2, g'_1, g'_2 \in H^0(\mathbb{P}^n, \mathcal{O}(1))$  and two linearly independent equations

$$\begin{aligned} g_1f_1 + g_2f_2 &= 0 \\ g'_1f_1 + g'_2f_2 &= 0, \end{aligned}$$

with both sets  $(g_1, g_2), (g'_1, g'_2)$  linearly independent. From the first equation it follows that  $f_1 = g_2h$  and  $f_2 = -g_1h$ , for some common factor  $h$ . Applying this to the second equation, we find  $g'_1g_2 = g'_2g_1$ , hence  $g'_1 = \alpha g_1$  and  $g'_2 = \alpha g_2$  for some scalar  $\alpha$ , a contradiction.  $\square$

**Corollary 2.6.** *Let  $k = d + 1$ , let  $T \subset |\mathcal{O}(d)|$  be a line spanned by  $f_1$  and  $f_2$ . The type  $(2, 1, \dots, 1, 0, \dots, 0)$  occurs if and only if  $\deg(\text{gcd}(f_1, f_2)) \geq d - 1$ .  $\square$*

### 3 The cohomology class of the set of jumping lines

**Definition 3.1.** Let  $k \geq 1$  and  $(b_i)$  be a splitting type for  $V_k$ . We define the set  $Z_{(b_i)}$  of all points  $t \in \mathbb{G}r(1, |\mathcal{O}(d)|)$  such that  $V_{k,t}$  has splitting type  $(b_i)$ . For the set of points  $t$  where  $V_{k,t}$  has generic splitting type we also write  $Z_{\text{gen}}$ , and define the *set of jumping lines*  $Z := \mathbb{G}r(1, |\mathcal{O}(d)|) \setminus Z_{\text{gen}}$ .

Now let  $k = d + 1$ . By Corollary 2.6,  $Z$  is the subvariety given as the image of the finite, generically injective multiplication map

$$\varphi: \mathbb{G}r(1, |\mathcal{O}(1)|) \times |\mathcal{O}(d-1)| \rightarrow \mathbb{G}r(1, |\mathcal{O}(d)|)$$

sending the tuple  $((sg_1 + tg_2)_{(s:t) \in \mathbb{P}^1}, h)$  to the line  $(shg_1 + thg_2)_{(s:t) \in \mathbb{P}^1}$ .

To perform calculations in the Chow ring  $A$  of  $\mathbb{G}r(1, |\mathcal{O}(d)|)$ , we follow the conventions found in [EH16]. Let  $N := \dim H^0(\mathcal{O}(d)) = \binom{n+d}{n}$ . For  $N - 2 \geq a \geq b$ , we have the Schubert cycle

$$\Sigma_{a,b} := \{T \in \mathbb{G}r(1, |\mathcal{O}(d)|) : T \cap H \neq \emptyset, T \subseteq H'\},$$

where  $(H \subset H')$  is a general flag of linear subspaces of dimension  $N - a - 2$  resp.  $N - b - 1$  in the projective space  $|\mathcal{O}(d)|$ . The ring  $A$  is generated by the Schubert classes  $\sigma_{a,b}$  of the cycles  $\Sigma_{a,b}$ . The class  $\Sigma_{a,b}$  has codimension  $a + b$ , and we use the convention  $\sigma_a := \sigma_{a,0}$ .

*Proof of Theorem 1.1.* We have  $\dim Z = n + 1 + \binom{d-1+n}{n}$  since  $Z$  is the image of the generically injective map  $\varphi$ .

Let  $Q \subset |\mathcal{O}(d)|$  be the image of the multiplication map

$$f: |\mathcal{O}(1)| \times |\mathcal{O}(d-1)| \rightarrow |\mathcal{O}(d)|.$$

The map  $f$  is birational on its image, since a general point of  $Q$  has the form  $gh$  with  $h$  irreducible of degree  $d - 1$ . The Chow group  $A^{\text{codim } Z}$  is generated by the classes  $\sigma_{a',b'}$  with  $N - 2 \geq a' \geq b' \geq \lfloor \frac{\text{codim } Z}{2} \rfloor$  and  $a' + b' = \text{codim } Z$ , while the complementary group  $A^{\dim Z}$  is generated by the classes  $\sigma_{\dim Z - b, b}$  with  $b \in 0, \dots, \lfloor \frac{\dim Z}{2} \rfloor$ . Write

$$[Z] = \sum_{a',b'} \alpha_{a',b'} \sigma_{a',b'}.$$

We have  $\sigma_{a',b'} \sigma_{a,b} = 1$  if  $b' - b = \lfloor \frac{\text{codim } Z}{2} \rfloor$  and 0 else. Hence, multiplying the above equation with the complementary classes  $\sigma_{a,b}$  and taking degrees gives

$$\alpha_{a',b'} = \deg([Z] \cdot \sigma_{a,b}).$$

Using Giambelli's formula  $\sigma_{a,b} = \sigma_a \sigma_b - \sigma_{a+1} \sigma_{b-1}$  [EH16, Prop. 4.16], we reduce to computing  $\deg([Z] \cdot \sigma_a \sigma_b)$  for  $0 \leq b \leq \lfloor \frac{\dim Z}{2} \rfloor$ . By Kleiman transversality, we have

$$\deg([Z] \cdot \sigma_a \sigma_b) = |\{T \in Z : T \cap H \neq \emptyset, T \cap H' \neq \emptyset\}|,$$

where  $H$  and  $H'$  are general linear subspaces of  $|\mathcal{O}(d)|$  of dimension  $N - a - 2$  and  $N - b - 2$ , respectively.

To a point  $p = g_p h_p \in Q$  with  $g_p \in |\mathcal{O}(1)|$  and  $h_p \in |\mathcal{O}(d-1)|$ , associate a closed reduced subscheme  $\Lambda_p \subset Q$  containing  $p$  as follows. If  $h_p$  is irreducible, let  $\Lambda_p$  be the image of the linear embedding  $|\mathcal{O}(1)| \times \{h_p\} \rightarrow |\mathcal{O}(d)|$  given by  $g \mapsto gh_p$ .

If  $h_p$  is reducible, define the subscheme  $\Lambda_p$  as the union  $\bigcup_h \text{im}(|\mathcal{O}(1)| \times \{h\} \rightarrow |\mathcal{O}(d)|)$ , where  $h$  ranges over the (up to multiplication by units) finitely many divisors of  $p$  of degree  $d - 1$ .

Note that for all points  $p$ , the spaces  $\text{im}(|\mathcal{O}(1)| \times \{h\} \rightarrow |\mathcal{O}(d)|)$  meet exactly at  $p$ .

By the definition of  $Z$ , all lines  $T \in Z$  lie in  $Q$ . Furthermore, if  $T$  meets the point  $p$ , then  $T \subseteq \Lambda_p$ . For  $H \subseteq |\mathcal{O}(d)|$  a linear subspace of dimension  $N - a - 2$ , define  $Q' := H \cap Q$ . For general  $H$ , the subscheme  $Q'$  is a smooth subvariety of dimension  $b - n + 1$  such that for a general point  $p = gh$  of  $Q'$  with  $h \in |\mathcal{O}(d)|$ , the polynomial  $h$  is irreducible.

Next, we consider the case  $n = 2$  or  $\dim Z$  odd.

**Claim 3.1.1.** For general  $H$ , for each point  $p \in Q'$  we have  $\Lambda_p \cap H = \{p\}$ .

*Proof.* Let  $\mathcal{H}$  denote the Grassmannian  $\text{Gr}(\dim H + 1, N)$ . Define the closed subset  $X \subseteq Q \times \mathcal{H}$  by

$$X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}.$$

The fibers of the induced map  $X \rightarrow \mathcal{H}$  have dimension at least one. Hence, to prove that the desired condition on  $H$  is an open condition, it suffices to prove  $\dim(X) \leq \dim(\mathcal{H})$ .

The fiber of the map  $X \rightarrow Q$  over a point  $p$  consists of the union of finitely many closed subsets of the form  $X'_p = \{H \in \mathcal{H} : \dim(H \cap \Lambda'_p) \geq 1\}$ , where  $\Lambda'_p \simeq \mathbb{P}^n \subseteq |\mathcal{O}(d)|$  is one of the components of  $\Lambda_p$ . The space  $X'_p$  is a Schubert cycle

$$\Sigma_{\dim Q - b, \dim Q - b} = \{H \in \text{Gr}(\dim H + 1, N) : \dim(H \cap H_{n+1}) \geq 2\},$$

with  $H_{n+1}$  an  $(n+1)$ -dimensional subspace of  $H^0(\mathcal{O}(d))$ . The codimension of the cycle is  $2(\dim Q - b)$ , hence also  $\text{codim}(X_p) = 2(\dim Q - b)$ . Finally, we have  $\dim(\mathcal{H}) - \dim(X) = \text{codim}(X_p) - \dim(Q) = \dim Q - 2b$ .

If  $\dim Z$  is odd, then  $\dim Q - 2b \geq \dim Q - \dim Z + 1 = 3 - n \geq 0$ . If  $n = 2$ , we instead estimate  $\dim Q - 2b \geq \dim Q - \dim Z = 2 - n \geq 0$ .  $\blacksquare$

Next, let

$$\Lambda := \bigcup_{p \in Q'} \Lambda_p = f(|\mathcal{O}(1)| \times \text{pr}_2 f^{-1}(Q'))$$

and

$$\Lambda'' := |\mathcal{O}(1)| \times \text{pr}_2 f^{-1}(Q').$$

By the choice of  $H$ , the map  $f^{-1}(Q') \rightarrow Q'$  is birational and the map  $f^{-1}(Q') \rightarrow \text{pr}_2 f^{-1}(Q')$  is even bijective. It follows that  $\Lambda''$  and hence  $\Lambda$  have dimension  $b + 1$ .

The intersection of  $\Lambda$  with a general linear subspace  $H'$  of dimension  $N - b - 2$  is a finite set of points. For each point  $p \in Q'$ , the linear subspace  $H'$  intersects each component  $\Lambda'_p$  of  $\Lambda_p$  in at most one point. For each point  $p' \in H' \cap \Lambda$  there exists a unique  $p$  such that  $p' \in \Lambda_p$ .

The only line  $T \in Z$  meeting both  $p$  and  $H'$  is the one through  $p$  and  $p'$ . If the intersection  $H' \cap \Lambda_p$  is empty, then there will be no line meeting  $p$  and  $H'$ . Hence,  $\deg([Z] \cdot \sigma_a \sigma_b)$  is the number of intersection points of  $\Lambda$  with a general  $H'$ .

Finally, the pre-image  $f^{-1}(Q') = f^{-1}(H)$  is smooth for a general  $H$  by Bertini's Theorem. If  $\zeta$  is the class of a hyperplane section of  $|\mathcal{O}(d)|$  we have  $f^*(\zeta) = \alpha + \beta$ , where  $\alpha$  and  $\beta$  are classes of hyperplane sections of  $|\mathcal{O}(1)|$  and  $|\mathcal{O}(d)|$ , respectively. Since  $\text{pr}_2$  and  $f$  have degree one, we compute

$$[\Lambda''] = [\text{pr}_2^{-1} \text{pr}_2 f^{-1}(H)] = \text{pr}_2^* \text{pr}_{2,*} f^*[H] = \binom{\text{codim } H}{n} \beta^{\text{codim } H - n}.$$

Hence, by the push-pull formula:

$$\deg([\Lambda] \cdot H') = \deg([\Lambda''] \cdot (\alpha + \beta)^{\text{codim } H'}) = \binom{\text{codim } H}{n} \binom{\text{codim } H'}{n} = \binom{a+1}{n} \binom{b+1}{n}.$$

We then use Giambelli's formula to obtain Equation (1.1).

In case  $n = 3$  and  $\dim Z$  even, we show that for  $b = \dim Z/2$  we have  $\deg([Z] \cdot \sigma_{b,b}) = 0$ . In this case, the hyperplanes  $H$  and  $H'$  have the same dimension  $N - b - 2$ .

For  $p \in Q$ , the set  $\Lambda_p$  is defined as before.

**Claim 3.1.2.** for general  $H$  of dimension  $N - b - 2$ , we have  $\dim(\Lambda_p \cap H) = 1$ .

*Proof.* Define as before the closed subset  $X \subseteq Q \times \mathcal{H}$  by

$$X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}.$$

The generic fiber of the projection map  $\varphi: X \rightarrow \mathcal{H}$  is one-dimensional, hence we have  $\dim \varphi(X) = \dim(X) - 1 = \dim \mathcal{H}$ . The last equation holds with  $n = 3$  and  $2b = \dim Z$ . Hence for all  $H \in \mathcal{H}$  we have  $\dim(\Lambda_p \cap H) \geq 1$ .

On the other hand, the equality  $\dim(\Lambda_p \cap H) = 1$  is attained by some, and hence by a general,  $H$ . Indeed, Define the closed subset  $X \subseteq Q \times \mathcal{H}$  by

$$X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}.$$

By a similar argument as before, one needs to show that  $\dim(\mathcal{H}) - \dim(X) + 1 \geq 0$ . The fiber  $X_p$  is a Schubert cycle of codimension  $3(\dim Q - b + 1)$ . Lastly, a computation shows  $\dim(\mathcal{H}) - \dim(\tilde{X}) + 1 = \text{codim}(\tilde{X}_p) - \dim(Q) + 1 = \frac{1}{2}(2 \dim Q + 18 - 5n) \geq 0$ . ■

Now, define  $\Lambda''$  as above. We have  $\dim \Lambda'' = \dim |\mathcal{O}(1)| + \dim \text{pr}_2 f^{-1}(Q') = b$ . Since  $f$  is generically of degree one, we still have  $\dim \Lambda'' = \Lambda$ , hence  $\dim \Lambda + \dim H' = N - 2 < \dim |\mathcal{O}(d)|$ . It follows that a generic  $H'$  does not meet any of the lines  $T \subset Z$ , hence  $\sigma_b \sigma_b \cdot [Z] = 0$ .  $\square$

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