Verlinde Bundles of Families of Hypersurfaces

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1 Introduction

Let $\pi\colon X \to S$ be a flat proper morphism of schemes, $\mathcal{L}$ an ample line bundle on X. The line bundle $\mathcal{L}$ and its powers $\mathcal{L}^{\otimes k}$ give information about how to embed the members X_s of the family π into the projective space, and how these embeddings vary along S. To better understand the family π, one can study the pushforwards $V_k := \pi_*(\mathcal{L}^{\otimes k})$, for $k \geq 1$. In good situations, V_k is a vector bundle on S and we have $V_k|_s = H^0(X_s, \mathcal{L}^{\otimes k})$ for $s \in S$. The V_k are then called the *Verlinde bundles* of the family π.

This thesis examines the situation where π is the universal family of hypersurfaces of degree d in $\mathbb{P}^n$, where $n > 1$. The Verlinde bundles are then defined on $|\mathcal{O}(d)|$, which is a projective space of dimension $\binom{d+n}{n} - 1$.

We focus on the restriction $V_k|_T$ of the Verlinde bundles to lines $T \subset |\mathcal{O}(d)|$, asking which isomorphism types of bundles on $\mathbb{P}^1$ can occur as some such restriction, and how these types depend on T. For this, we study the associated subsets of the Grassmannian on lines in $|\mathcal{O}(d)|$ consisting of lines T such that $V_k|_T$ has a fixed isomorphism type. Lastly, we investigate some global statements about V_k, sometimes using its restriction to lines.

1.1 Notation and conventions

Throughout, k will denote an algebraically closed field, but we omit it from most notation. The letter k will also denote a natural number.

For natural numbers d and n, we write I_d for a tuple of non-negative integers of the form $(i_0, \dots, i_n)$ with $\sum i_j = d$. Thus for example a tuple ranging over the I_d will have $\binom{n+d}{n}$ entries.

We fix names for the homogeneous coordinates of various projective spaces: for the coordinates of $\mathbb{P}^1$ we write s and t, for $\mathbb{P}^n$ we write x_i, and for the coordinates of $|\mathcal{O}(d)|$ we take α_{I_d}, where we think of α_{I_d} as corresponding to $x^{I_d} \coloneqq \prod_i x_i^{(I_d)_i}$.

For a fiber product $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ and sheaves $\mathcal{F}$ and $\mathcal{G}$ on X resp. Y, we write $\mathcal{F} \boxtimes \mathcal{G} \coloneqq p^*\mathcal{F} \otimes q^*\mathcal{G}$.

1.2 Aknowledgements

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2 Universal Families of Extensions

Let X and S be Noetherian schemes over a field k. Let $f\colon X \to S$ be a flat, projective morphism, and let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules, flat over $\mathcal{O}_S$.

Recall that an element $\xi \in \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$ corresponds to an equivalence class of short exact sequences, or *extensions*, of the form

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0,$$

where two such sequences are equivalent if there exists an isomorphism between them that induces the identity on $\mathcal{F}$ and $\mathcal{G}$. The set of these equivalence classes can be given the structure of an $H^0(S, \mathcal{O}_S)$-module, see for example [Wei94, 3.4]. This correspondence is functorial in both arguments, and preserves the $H^0(S, \mathcal{O}_S)$-module structure.

Explicitely, the sum of two elements of Ext^1 corresponds to the Baer sum of the associated extensions, while the multiplication of an extension as above by a scalar $a \in H^0(S, \mathcal{O}_S)$ is given by the pullback sequence along the map $\mathcal{F} \xrightarrow{a} \mathcal{F}$.

In this section, we ask when it is possible to construct an S-scheme V and a universal extension on $X \times_S V$. This is found to be true for $\mathcal{F}, \mathcal{G}$ locally free and $S = \operatorname{Spec}(k)$. For the more general situation, the article [Lan83] turns to the moduli problem of classifying relative extensions of sheaves and applies it to the global situation.

Remark 2.1. Let $\varphi\colon X_1 \to X_2$ be a morphism of schemes, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be $\mathcal{O}_{X_1}$-modules. The Grothendieck spectral sequence [Vak17, Theorem 23.3.5] specializes to the Leray spectral sequence $E_2^{p,q} = H^p(X_2, R^q\varphi_*\mathcal{F}_1) \Rightarrow H^{p+q}(X_1, \mathcal{F}_1)$ and the local-to-global Ext spectral sequence $E_2^{p,q} = H^p(X_1, \mathcal{E}xt^q(\mathcal{F}_1, \mathcal{F}_2)) \Rightarrow \operatorname{Ext}^{p+q}(\mathcal{F}_1, \mathcal{F}_2)$. The first few terms of the associated exact sequences in lower degrees are

$$0 \to H^1(X_2, \varphi_*\mathcal{F}_1) \to H^1(X_1, \mathcal{F}_1) \to H^0(X_2, R^1\varphi_*\mathcal{F}_1) \tag{2.1}$$

and

$$0 \to H^1(X_2, \mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)) \to \operatorname{Ext}^1(\mathcal{F}_1, \mathcal{F}_2) \to H^0(X_2, \mathcal{E}xt^1(\mathcal{F}_1, \mathcal{F}_2)). \tag{2.2}$$

Proposition 2.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be locally free, and let $V := \mathbb{V}(\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})^\vee)$. There exists an extension*

$$\xi_{\mathrm{univ}}\colon \quad 0 \to \mathrm{pr}_1^*\mathcal{G} \to \mathcal{E} \to \mathrm{pr}_1^*\mathcal{F} \to 0$$

over $X \times_k V$ such that for all Noetherian k-schemes Y, the map

$$\operatorname{Mor}_k(Y, V) \to \operatorname{Ext}^1_{X_Y}(\mathcal{F}_Y, \mathcal{G}_Y)$$

defined by $\alpha \mapsto (\mathrm{id}_X \times \alpha)^*\xi_{\mathrm{univ}}$ is a bijection, functorial in Y. In particular, pulling back ξ_{univ} gives a bijection $\operatorname{Mor}_k(\operatorname{Spec}(k), V) \xrightarrow{\sim} \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})$.

Proof. We denote the maps in the cartesian square $Y \times X$ as follows:

$$\begin{array}{ccc} Y \times X & \xrightarrow{\mathrm{pr}_1} & Y \\ {\scriptstyle \mathrm{pr}_2}\downarrow & \times & \downarrow{\scriptstyle s_2} \\ X & \xrightarrow[s_1]{} & k \end{array}$$

We find functorial isomorphisms

$$\begin{aligned} \operatorname{Mor}_k(Y, V) &\simeq H^0(\mathcal{O}_Y \otimes \operatorname{Ext}^1(\mathcal{F}, \mathcal{G})) && (2.3)\\ &\simeq H^0(s_2^* H^1(\mathcal{H}om(\mathcal{F}, \mathcal{G}))) && (2.4)\\ &\simeq H^0(R^1\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\mathcal{H}om(\mathcal{G}, \mathcal{F}))) && (2.5)\\ &\simeq H^0(R^1\mathrm{pr}_{2,*}(\mathcal{H}om(\mathcal{F}_Y, \mathcal{G}_Y))) && (2.6)\\ &\simeq H^1(\mathcal{H}om(\mathcal{G}_Y, \mathcal{F}_Y)) && (2.7)\\ &\simeq \operatorname{Ext}^1_{X_Y}(\mathcal{F}_Y, \mathcal{G}_Y). && (2.8) \end{aligned}$$

The required universal extension is then the image of $\mathrm{id} \in \operatorname{Mor}_k(V, V)$ in $\operatorname{Ext}^1_{X_Y}(\mathcal{F}_Y, \mathcal{G}_Y)$.

The isomorphism (2.3) comes from the universal property of $\mathbb{V}(\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{G})^\vee)$. The isomorphisms (2.4) and (2.8) come from the sequence (2.2), whose third term is zero since $\mathcal{F}$ and $\mathcal{G}$ are locally free. We have (2.5) by the Cohomology and Base Change Theorem [Vak17, 28.1.6], and (2.6) since $\mathcal{F}$ and $\mathcal{G}$ are locally free. For the isomorphism (2.7), we use the sequence (2.1), whose third term is found to be zero after applying the Cohomology and Base Change Theorem. □

Definition 2.3. (i) The *i-th relative Ext module* $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ is the image of $\mathcal{G}$ under the i-th right-derived functor of $f_*\mathcal{H}om(\mathcal{F},-)\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_S}$.

(ii) For $s \in S$, define the homomorphism

$$\Phi_s = \Phi_{s,\mathcal{F},\mathcal{G}}\colon \operatorname{Ext}^1_X(\mathcal{F},\mathcal{G}) \to \operatorname{Ext}^1_{X_s}(\mathcal{F}_s,\mathcal{G}_s)$$

by restricting extensions of $\mathcal{F}$ by $\mathcal{G}$ to the fiber X_s. This is well-defined, since $\mathcal{F}$ is flat over S.

(iii) A *family of extensions* of $\mathcal{F}$ by $\mathcal{G}$ over S is a family

$$\xi_s \in \operatorname{Ext}^1_{X_s}(\mathcal{F}_s,\mathcal{G}_s), \quad s \in S$$

such that there exists an open covering $\mathfrak{U}$ of S and for all $U \in \mathfrak{U}$ an extension $\xi_U \in \operatorname{Ext}^1_{f^{-1}(U)}(\mathcal{F}_U,\mathcal{G}_U)$ with $\Phi_{s,\mathcal{F}_U,\mathcal{G}_U}(\xi_U) = \xi_s$ for all $s \in S$. Such a family is *globally defined* if we can take $\mathfrak{U} = \{S\}$.

Remark 2.4. If S is affine, then we have $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G}) = \operatorname{Ext}^i_X(\mathcal{F},\mathcal{G})\tilde{}$.

Proposition 2.5. *Let $g\colon Y \to S$ be a morphism of Noetherian schemes. For all $i \geq 0$ there exists a canonical base change homomorphism*

$$\tau^i_g\colon g^*\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G}) \to \mathcal{E}xt^i_{f_Y}(g_X^*\mathcal{F}, g_X^*\mathcal{G}).$$

Furthermore, if g is flat, then τ^i_g is an isomorphism for all $i \geq 0$.

Proof. See [Lan83, Prop. 1.3] □

Definition 2.6. We say that $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ *commutes with base change* if for all morphisms of Noetherian schemes $g\colon Y \to S$, the base change homomorphism τ^i_g is an isomorphism.

Proposition 2.7. *Let $s \in S$ be a point such that τ^i_s is surjective. Then there exists an open neighborhood U of s such that $\tau^i_{s'}$ is an isomorphism for all $s' \in U$. Furthermore, the homomorphism τ^{i-1}_s is surjective if and only if $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ is locally free on an open neighborhood of s.*

Proof. See [Lan83, Thm. 1.4] □

Remark 2.8. (i) If τ^i_s is an isomorphism for all $s \in S$, then $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ commutes with base change.

(ii) From Proposition 2.7 we conclude that if $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ commutes with base change for $i = 0, 1$, then $\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})$ is locally free.

(iii) In case S is reduced, if $\mathcal{E}xt^1_f(\mathcal{F},\mathcal{G})$ is locally free then $\mathcal{E}xt^i_f(\mathcal{F},\mathcal{G})$ commutes with base change for $i = 0, 1$.

Definition 2.9. Let $u\colon Y' \to Y$ be a morphism of Noetherian S-schemes.

(i) We define a functoriality map $H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) \to H^0(Y', \mathcal{E}xt^1_{f_{Y'}}(\mathcal{F}_{Y'}, \mathcal{G}_{Y'}))$ as the composition

$$\begin{aligned} H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) &\xrightarrow{1\otimes \mathrm{id}} H^0(Y', u^*\mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) \\ &\xrightarrow{H^0(\tau^1_u)} H^0(Y', \mathcal{E}xt^1_{f_{Y'}}(u^*_{X_Y}\mathcal{F}_{Y'}, u^*_{X_Y}\mathcal{G}_{Y'})). \end{aligned}$$

(ii) Given a family of extensions $\xi = (\xi_y)_{y\in Y}$ of $\mathcal{F}_Y$ by $\mathcal{G}_Y$ over Y, we set $(u^*\xi)_{y'} := u^*\xi_{u(y')}$ for every $y' \in Y'$. This defines a family $u^*\xi$ of extensions of $\mathcal{F}_{Y'}$ by $\mathcal{G}_{Y'}$ over Y'.

(iii) We define the functors

$$\begin{aligned} &E, E' \colon (\mathrm{NoethSch}/S) \to (\mathrm{Sets}); \\ &E(Y) := H^0(Y, \mathcal{E}xt^1_{f_Y}(\mathcal{F}_Y, \mathcal{G}_Y)), \\ &E'(Y) := \{\text{families of extensions of } \mathcal{F}_Y \text{ by } \mathcal{G}_Y \text{ over } Y\}. \end{aligned}$$

Remark 2.10. The Grothendieck spectral sequence for the sequence of functors

$$\mathrm{Mod}_{\mathcal{O}_X} \xrightarrow{\mathcal{H}om(\mathcal{F},-)} \mathrm{Mod}_{\mathcal{O}_X} \xrightarrow{f_*} \mathrm{Mod}_{\mathcal{O}_S}$$

is the spectral sequence with $E_2^{p,q} = H^p(S, \mathcal{E}xt^q_f(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}_X(\mathcal{F}, \mathcal{G})$. This gives the exact sequence

$$\begin{aligned} 0 \to &H^1(S, f_*\mathcal{H}om(\mathcal{F}, \mathcal{G})) \xrightarrow{\varepsilon} \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G}) \xrightarrow{\mu} H^0(S, \mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})) \\ \xrightarrow{d_2} &H^2(S, f_*\mathcal{H}om(\mathcal{F}, \mathcal{G})). \end{aligned} \tag{2.9}$$

The special case $f = \mathrm{id}$ gives Remark 2.1.

Proposition 2.11. *Suppose that S is reduced and $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})$ commutes with base change. Restricted to the category of reduced Noetherian S-schemes, the functors E and E' are isomorphic.*

Proof. See [Lan83, Prop. 2.3]. □

Proposition 2.12. *Suppose that $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Then the $\mathcal{O}_S$-module $\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})^\vee$ is locally free and the functor E is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})^\vee)$.*

Proof. See [Lan83, Prop. 3.1]. □

Corollary 2.13. *Suppose that S is reduced and $\mathcal{E}xt^i_f(\mathcal{F}, \mathcal{G})$ commutes with base change for $i = 0, 1$. Restricted to the category of reduced Noetherian S-schemes, the functor E' is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})^\vee)$.*

Corollary 2.14. *Suppose that for all Noetherian S-schemes Y we have*

$$H^i(Y, f_{Y,*}\mathcal{H}om_{X_Y}(\mathcal{F}_Y, \mathcal{G}_Y)) = 0$$

for $i = 1, 2$. The functor $Y \mapsto \mathrm{Ext}^1_{X_Y}(\mathcal{F}_Y, \mathcal{G}_Y)$ is representable by the S-scheme $\mathbb{V}(\mathcal{E}xt^1_f(\mathcal{F}, \mathcal{G})^\vee)$.

Proof. Use the sequence (2.9) and Proposition 2.12. □

Remark 2.15. As a special case of the above, we recover Proposition 2.2.

Remark 2.16. The article [Lan83] continues to define a (projectivized) version of the problem, so that over $\mathrm{Spec}(k)$, the scheme $\mathbb{P}(\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G})^\vee)$ parametrizes the equivalence classes of nonsplit extensions of $\mathcal{F}$ by $\mathcal{G}$, modulo the action of $k^\times$. See also [HL10, Example 2.1.12].

3 Specialization of Vector Bundles on $\mathbb{P}^1$

Let V be a vector bundle on $\mathbb{P}^1$. The Birkhoff–Grothendieck theorem says that V can be written as a direct sum $V = \bigoplus_i \mathcal{O}(b_i)$, with a unique tuple (b_i) of integers. The tuple (b_i) is also called the *splitting type* of V. To study a vector bundle W on a projective space $\mathbb{P}^m$, one can examine the restriction of W to lines $T \subset \mathbb{P}^m$, and ask how the splitting type of $W|_T$ varies with T. Which splitting behavior should one expect for general T? To answer this question, we look at families of vector bundles over $\mathbb{P}^1$ and ask about their general and special members. Which pairings of a general and a special splitting type are possible in a family? This section makes this question precise using the notion of specialization and provides an answer.

Definition 3.1. Let V and V' be vector bundles on a projective k-scheme X. We say that V *specializes* to V' if there exists an affine k-scheme Y, spectrum of a discrete valuation ring, with generic point η and closed point η_0, and a vector bundle W on $Y \times X$ such that $W|_{\eta \times X} \simeq \kappa(\eta) \boxtimes V$ and $W|_{\eta_0 \times X} \simeq \kappa(\eta_0) \boxtimes V'$.

Remark 3.2. If V specializes to V' and W specializes to W', then $V \oplus W$ specializes to $V' \oplus W'$.

Remark 3.3. Specialization is transitive for $X = \mathbb{P}^1$: if V specializes to V' and V' specializes to V'', then V specializes to V'', see e.g. [Ram83, Cor. 6.14].

Remark 3.4. This definition reflects specialization of points in the moduli stack Vect_X of vector bundles over X, where e.g. the bundles V and $\kappa(\eta) \boxtimes V$ define the same point. The stack is locally Noetherian, hence discrete valuation rings suffice. This notion generalizes the notion of specialization of points on a scheme, see e.g. [ÉGA II, Prop. 7.1.9].

Remark 3.5. Let
$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{g} \mathcal{G} \to 0$$
be a short exact sequence of coherent sheaves over a k-scheme X and let $\xi \in \operatorname{Ext}^1(\mathcal{G}, \mathcal{F})$ be the corresponding element. If $a \in H^0(X, \mathcal{O}_X^\times)$, then the element $a\xi$ corresponds to the sequence
$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{a^{-1}g} \mathcal{G} \to 0.$$

Example 3.6. The vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on $\mathbb{P}^1$ specializes to $\mathcal{O} \oplus \mathcal{O}(2)$. This can be seen as follows. The elements of $\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O})$ correspond to extensions of the form
$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(2) \to 0$$
up to equivalence. The zero element corresponds to the split extension $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$. Note that all such extensions must have $\mathcal{E}$ locally free. Considering the formulae for ranks and determinants of the components of the sequence, we see that the nonsplit extensions must have $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$. Furthermore, we have
$$\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O}) = \operatorname{Ext}^1(\mathcal{O}, \mathcal{O}(-2)) = H^1(\mathcal{O}(-2)) = k.$$
By Proposition 2.2 and using $\mathbb{V}(\operatorname{Ext}^1(\mathcal{O}(2), \mathcal{O})^\vee) \simeq \mathbb{A}^1$, there exists an extension of the form
$$0 \to \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O} \to \mathcal{E}_{\text{univ}} \to \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}(2) \to 0$$
on $\mathbb{A}^1 \times \mathbb{P}^1$ such that for $\xi \in \mathbb{A}^1$ generic we have $\mathcal{E}_{\text{univ}}|_{\xi \times \mathbb{P}^1} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ and for $\xi = 0$ we have $\mathcal{E}_{\text{univ}}|_{0 \times \mathbb{P}^1} \simeq \mathcal{O} \oplus \mathcal{O}(2)$. Since $\mathcal{E}_{\text{univ}}$ is locally free, $\mathcal{O}(1) \oplus \mathcal{O}(1)$ specializes to $\mathcal{O}(2) \oplus \mathcal{O}$.

Remark 3.7. Let $b_1, \ldots, b_m$ be non-negative integers, let $a \coloneqq \sum b_i$, and let s, t denote the homogeneous coordinates on $\mathbb{P}^1$. The sequence
$$0 \to \mathcal{O}^{m-1} \xrightarrow{f} \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \xrightarrow{g} \mathcal{O}(a) \to 0$$
with
$$f = \begin{pmatrix} s^{b_1} & & & \\ t^{b_2} & s^{b_2} & & \\ & t^{b_3} & \ddots & \\ & & \ddots & s^{b_{m-1}} \\ & & & t^{b_m} \end{pmatrix}$$
and
$$g = \begin{pmatrix} -t^{a-b_1} & s^{b_1} t^{a-b_1-b_2} & \cdots & (-1)^m s^{b_1+\cdots+b_{m-1}} t^{a-b_1-\cdots-b_m} \end{pmatrix}$$
is exact.

Proposition 3.8. *Let $b_1, \ldots, b_m$ be non-negative integers and π a partition of the set $\{1, \ldots, m\}$. For a set of indices $I \in \pi$, let $b'_I \coloneqq \sum_{i \in I} b_i$. Then the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i)$ on $\mathbb{P}^1$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I) \oplus \mathcal{O}^{\oplus m - |\pi|}$.*

Proof. By Remark 3.2 it suffices to prove the special case $\pi = \{\{1, \dots, m\}\}$. In other words, we prove that if $a = \sum b_i$, then $\bigoplus \mathcal{O}(b_i)$ specializes to $\mathcal{O}(a) \oplus \mathcal{O}^{m-1}$. By Remark 3.7, there exists a representative $\xi \in \operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ of an exact sequence of the form

$$0 \to \mathcal{O}^{m-1} \to \mathcal{O}(b_1) \oplus \cdots \oplus \mathcal{O}(b_m) \to \mathcal{O}(a) \to 0. \tag{3.1}$$

By Remark 3.5, scalar multiplication by $\lambda \neq 0$ does not change the isomorphism class of the middle term of the sequence, hence there exists a one-dimensional subspace $k \hookrightarrow \operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})$ such that each nonzero element corresponds to an exact sequence of the form (3.1). Consider the associated closed embedding $\alpha \colon \mathbb{A}^1 \to \mathbb{V}(\operatorname{Ext}^1(\mathcal{O}(a), \mathcal{O}^{\oplus m-1})^\vee)$ and let $\mathcal{E}$ be the universal extension from Proposition 2.2. Then the vector bundle $(\mathrm{id}_{\mathbb{P}^1} \times \alpha)^* \mathcal{E}$ on $\mathbb{P}^1 \times \mathbb{A}^1$ realizes the required specialization. □

Remark 3.9. By twisting the exact sequence (3.1) in the proof of Proposition 3.8 and using the same argument, we see that for every integer n and with b_i, π, and b_I as above, the vector bundle $\bigoplus_{i=1}^m \mathcal{O}(b_i + n)$ specializes to $\bigoplus_{I \in \pi} \mathcal{O}(b'_I + n) \oplus \mathcal{O}(n)^{\oplus m - |\pi|}$.

Definition 3.10. For tuples (b_i) and (b'_i) of the same size r, we define the expression $(b'_i) \geq (b_i)$ to mean

$$\sum_{i=1}^s b'_i \geq \sum_{i=1}^s b_i \text{ for all } s = 1, \dots, r.$$

Proposition 3.11. *Let $b = (b_i)_{i=1}^m$ and $b' = (b'_i)_{i=1}^m$ be tuples of integers such that $\sum b_i = \sum b'_i$ and $b \leq b'$ in the sense of Definition 3.10. Then $\bigoplus \mathcal{O}(b_i)$ specializes to $\bigoplus \mathcal{O}(b'_i)$.*

Proof. Since $b \leq b'$, there exists a sequence of tuples $b = b^{(0)}, b^{(1)}, \dots, b^{(n)} = b'$ such that for all $k = 1, \dots, n$, the tuple $b^{(k)}$ differs from $b^{(k-1)}$ only by two entries

$$b_i^{(k)} = b_i^{(k-1)} - 1, \quad b_j^{(k)} = b_j^{(k-1)} + 1.$$

By Remark 3.9, $\mathcal{O}(b_i^{(k)}) \oplus \mathcal{O}(b_j^{(k)})$ specializes to $\mathcal{O}(b_i^{(k-1)}) \oplus \mathcal{O}(b_j^{(k-1)})$, hence $b^{(k-1)}$ specializes to $b^{(k)}$. By transitivity of specialization, b specializes to b'. □

Remark 3.12. In fact, if $\bigoplus \mathcal{O}(b_i)$ specializes to $\bigoplus \mathcal{O}(b'_i)$, then $(b_i) \leq (b'_i)$. The missing implication is proven e.g. in [Sha76, Thm. 3]. There, the statement is expressed in terms of the Harder–Narasimhan polygon HNP of a vector bundle E. In our case, if $E \simeq \bigoplus \mathcal{O}(b_i)$ is a vector bundle on $\mathbb{P}^1$, then HNP is the polygon under the graph of the function $\{0, \dots, \operatorname{rk} E\} \to \mathbb{N}_{\geq 0}$ given by $s \mapsto \sum_{i=1}^s b_i$. Thus, $\bigoplus \mathcal{O}(b_i)$ specializes to $\bigoplus \mathcal{O}(b'_i)$ if and only if the HNP of the first bundle lies below the HNP of the second.

Thus, the most general splitting type among bundles of fixed rank r and degree c_1 is the uniquely determined type of the form $(b+1, \dots, b+1, b, \dots, b)$. The integer b is determined by the equation $c_1 = br + a$, with $a < r$ becoming the number of occurences of $b + 1$.

4 General Facts about the Verlinde Bundles

This section introduces the main objects of study of this thesis, the Verlinde bundles of the universal family of hypersurfaces of degree d in $\mathbb{P}^n$. These were already studied as an example of Verlinde bundles of polarized families in the thesis [Hem15]. We reproduce the fundamental results about the Verlinde bundles, such as the fact they really are vector bundles, and introduce a presentation that we will employ throughout the rest of this work.

Throughout the rest of the thesis, we assume $n > 1$.

Definition 4.1. Let $\pi\colon \mathfrak{X} \to |\mathcal{O}_{\mathbb{P}^n}(d)|$ be the universal family of hypersurfaces of degree d in $\mathbb{P}^n$. Let $\mathcal{L}$ be the restriction to $\mathfrak{X}$ of the bundle $\mathcal{O}(1) \boxtimes \mathcal{O}$ under the inclusion $\mathfrak{X} \subseteq \mathbb{P}^n \times |\mathcal{O}(d)|$.

Proposition 4.2. *Let $k \geq 1$. The following statements hold:*

(i) *If $q \in |\mathcal{O}(d)|$ then $h^0(\mathfrak{X}_q, \mathcal{L}^{\otimes k}|_q) = \binom{k+n}{n} - \binom{k+n-d}{n}$. In particular, this number is independent of the point q.*

(ii) *The sheaf $\pi_*\mathcal{L}^{\otimes k}$ is locally free of rank $\binom{k+n}{n} - \binom{k+n-d}{n}$.*

(iii) *For all cartesian diagrams of the form*

$$\begin{array}{ccc} \mathfrak{X}_Z & \longrightarrow & \mathfrak{X} \\ {\scriptstyle \pi_Z}\downarrow & \times & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow[\rho]{} & |\mathcal{O}(d)| \end{array}$$

we have $\rho^*\pi_*\mathcal{L}^{\otimes k} \simeq (\pi_Z)_*\mathcal{L}_Z^{\otimes k}$.

Proof. The proof for the first statement is found in [Hem15, Proposition 4.1] and reproduced below. The others follow from Grauert's Theorem [Vak17, 28.1.5].

Let $X \coloneqq \mathfrak{X}_q$ be the hypersurface of degree d corresponding to the point q. We have $\mathcal{L}^{\otimes k}|_q = \mathcal{O}_{\mathbb{P}^n}(k)|_X$. Twisting the structure sequence

$$0 \to \mathcal{O}(-d) \to \mathcal{O} \to \mathcal{O}_X \to 0$$

on $\mathbb{P}^n$ with $\mathcal{O}(k)$ yields the short exact sequence

$$0 \to \mathcal{O}(k-d) \to \mathcal{O}(k) \to \mathcal{L}^{\otimes k}|_q \to 0.$$

The statement follows by taking global sections and using $H^1(\mathcal{O}(k-d), \mathbb{P}^n) = 0$ for $n > 1$. □

Definition 4.3. Let $k \geq 1$. The k-th *Verlinde bundle* of the family π is the vector bundle $V_k \coloneqq \pi_*\mathcal{L}^{\otimes k}$.

Proposition 4.4. *There exists a short exact sequence of vector bundles on* $|\mathcal{O}(d)|$

$$0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k-d)) \xrightarrow{M} \mathcal{O} \otimes H^0(\mathbb{P}^n, \mathcal{O}(k)) \to V_k \to 0. \tag{4.1}$$

The map M *is given by multiplication by* $\sum_I \alpha_I \otimes x^I \in H^0(|\mathcal{O}(d)|, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))$.

Proof. The proof, found in [Hem15, Proposition 4.2], is reproduced below.

The structure sequence of $\mathfrak{X}$ on $\mathbb{P}^n \times |\mathcal{O}(d)|$ is

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \boxtimes \mathcal{O}_{|\mathcal{O}(d)|}(-1) \to \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{|\mathcal{O}(d)|} \to \mathcal{O}_{\mathfrak{X}} \to 0,$$

the first map given by multiplication with $\sum_I \alpha_I \otimes x^I$. Twisting with $\mathcal{L}^{\otimes k} = \mathcal{O}(k) \boxtimes \mathcal{O}$, we get the exact sequence

$$0 \to \mathcal{O}(k-d) \boxtimes \mathcal{O}(-1) \to \mathcal{O}(k) \boxtimes \mathcal{O} \to \mathcal{L}^{\otimes k}|_X \to 0.$$

Applying the pushforward π_* we get the sequence (4.1), which is exact as

$$R^1\pi_*(\mathcal{O}(k-d) \boxtimes \mathcal{O}(-1)) = 0.$$

The description of the map M follows by the definition of the pushforward. □

Remark 4.5. For $k < d$, the sequence (4.1) shows that V_k is trivial. For $k = d$, the sequence (4.1) is the Euler sequence and $V_k = \mathcal{O}(-1)$.

5 Splitting Types for the Verlinde Bundles

Now that we have a presentation for V_k, we will try to exploit it as much as possible to extract information about the splitting types that can occur for the restriction of V_k to lines. We will see that the V_k are not uniform and that the expected generic splitting type is attained for $d \leq k < 2d$. We also demonstrate that the number of nonzero entries of the splitting type of $V_k|_T$ is known if we know two points in T.

5.1 Basics about splitting types

Definition 5.1. Let $T \subseteq |\mathcal{O}(d)|$ be a line, i.e. the closed subscheme defined as the image of a linear embedding $\mathbb{P}^1_K \to |\mathcal{O}(d)|$, with K an extension field of k. We call T a *pencil* of hypersurfaces. Its universal family is the scheme $\mathfrak{X}_{\mathbb{P}^1_K}$, which comes with the polarization $\mathcal{L}_{\mathbb{P}^1_K}$. The situation is summarized in the picture below:

$$\begin{array}{ccc} \mathfrak{X}_{\mathbb{P}^1_K} & \longrightarrow & \mathfrak{X} \\ \downarrow & \times & \downarrow \pi \\ \mathbb{P}^1_K & \longrightarrow & |\mathcal{O}(d)| \end{array}$$

Definition 5.2. On $\mathbb{P}^1$, we define the vector bundle $V_{k,T} := (\pi_{\mathbb{P}^1})_* \mathcal{L}_{\mathbb{P}^1}^{\otimes k}$. It is related to V_k by $V_k|_T = V_{k,T}$ using Proposition 4.2.

Remark 5.3. Let T be a pencil of hypersurfaces.

(i) The sequence (4.1) restricts to a sequence

$$0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k-d)) \to \mathcal{O} \otimes H^0(\mathbb{P}^n, \mathcal{O}(k)) \to V_{k,T} \to 0 \tag{5.1}$$

over $\mathbb{P}^1$.

(ii) By Proposition 4.2 and Proposition 4.4, the vector bundle $V_{k,T}$ has degree $\binom{k+n-d}{n}$ and rank $\binom{k+n}{n} - \binom{k+n-d}{n}$.

(iii) Let $V_{k,T} \simeq \bigoplus_i \mathcal{O}(b_i)$ be a splitting of $V_{k,T}$ over $\mathbb{P}^1$. By the sequence (5.1), we have $b_i \geq 0$.

Definition 5.4. Let $k \geq 1$.

(i) A *splitting type* for V_k is a non-increasing tuple $(b_1, \dots, b_{r^{(k)}})$ of non-negative integers with $r^{(k)} := \binom{k+n}{n} - \binom{k+n-d}{n}$ and $d^{(k)} := \sum b_i = \binom{k+n-d}{n}$.

(ii) The *generic splitting type* for V_k is the unique splitting type for V_k of the form $(b^{(k)}+1, \dots, b^{(k)}+1, b^{(k)}, \dots, b^{(k)})$.

(iii) Let E be a locally free sheaf on $\mathbb{P}^1$. The *splitting type* of E is the unique non-increasing tuple $(b_1, \dots, b_{r^{(k)}})$ such that $E \simeq \bigoplus_i \mathcal{O}(b_i)$.

Remark 5.5. Note that the degrees of $d^{(k)}$ and $r^{(k)}$ as polynomials in k are n and $n-1$, respectively. Hence, $b^{(k)} \to \infty$ for $k \to \infty$.

Proposition 5.6. *For $n \geq 2$, if $k \leq 2d$ then $b^{(k)} = 0$.*

Proof. We compute

$$\begin{aligned}
\frac{d^{(k)} + r^{(k)}}{d^{(k)}} &= \frac{(k+1)\cdots(k+n)}{(k-d+1)\cdots(k-d+n)} \\
&= \frac{(k-d+n+1)\cdots(k+n)}{(k-d+1)\cdots(k)} \\
&= (1+\frac{n}{k-d+1})(1+\frac{n}{k-d+2})\cdots(1+\frac{n}{k}) \\
&\geq (1+\frac{2}{k-d+1})\cdots(1+\frac{2}{k}) \\
&> 1 + d\frac{2}{k} \\
&\geq 2.
\end{aligned}$$

Hence, $d^{(k)} < r^{(k)}$. ∎

Definition 5.7. Let P denote the universal $\mathbb{P}^1$-bundle over the Grassmannian of lines $\mathrm{Gr}(2, H^0(\mathcal{O}(d))) = \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$, let $\varphi\colon P \to \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$ be the universal map and $p\colon P \to |\mathcal{O}(d)|$ the canonical projection.

$$\begin{array}{ccc} P & \xrightarrow{\quad p \quad} & |\mathcal{O}(d)| \\ \big\downarrow{\scriptstyle\varphi} & & \\ \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|) & & \end{array}$$

The mapping $t \mapsto P_t$ gives a canonical bijection between the points of $\mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$ and the pencils of hypersurfaces in $|\mathcal{O}(d)|$. For such t, we write $V_{k,t} := V_{k,p(P_t)}$.

5.2 Examples of Non-Uniform V_k

Example 5.8. A vector bundle V on a projective space $\mathbb{P}^m$ is called *uniform* if the splitting type of $V|_T$ of V does not depend on the choice of the line $T \subset \mathbb{P}^m$. Although the Verlinde bundles V_k are uniform for $k \leq d$, they are not uniform for $k > d$. For example, let $n = 2, k = 3, d = 2$. Then $V_k = \mathrm{coker}(M)$ with

$$M = \begin{pmatrix} \alpha_{00} & & \\ \alpha_{01} & \alpha_{00} & \\ \alpha_{02} & & \alpha_{00} \\ \alpha_{11} & \alpha_{01} & \\ \alpha_{12} & \alpha_{02} & \alpha_{01} \\ \alpha_{22} & & \alpha_{02} \\ & \alpha_{11} & \\ & \alpha_{12} & \alpha_{11} \\ & \alpha_{22} & \alpha_{12} \\ & & \alpha_{22} \end{pmatrix},$$

where α_{ij} is the coordinate function corresponding to the quadric $x_i x_j$. If T is a pencil of the form $(sf + tg)_{(s:t)\in\mathbb{P}^1}$ with $f = \sum_I \lambda_I x^I$ and $g = \sum_I \mu_I x^I$, then $V_k|_T = \mathrm{coker}(M_T)$, where M_T is obtained from M by the substitution $\alpha_{ij} \leftarrow \lambda_{ij} s + \mu_{ij} t$. Using Remark 3.7, we see that for $f = x_0^2$ and $g = x_1^2$ we have $\mathrm{coker}(M_T) = \mathcal{O}(3)^{\oplus 3}$, while for $f = x_0^2$ and $g = x_0 x_1$ we have $M|_T = \mathcal{O}(2) \oplus \mathcal{O}(1)$

Example 5.9. For $d = 3, n = 2, k = 5$, writing down M as above and trying out different monomials for f and g, one finds that the tuples $(3, 2, 1, 0_{12})$, $(2, 1_4, 0_{10})$, and $(1_6, 0_9)$ are possible splitting types of $V_k|_T$.

5.3 Attained splitting types

Lemma 5.10. *Let $\mathcal{E}$ be a free $\mathcal{O}_{\mathbb{P}^1}$-module of finite rank, and let*

$$0 \to \mathcal{E}' \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\psi} \mathcal{E}'' \to 0$$

be a short exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules. Given a splitting $\mathcal{E}'' = \mathcal{E}_1'' \oplus \mathcal{O}$, we may construct a splitting $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{O}$ such that the image of φ is contained in $\mathcal{E}_1$.

Proof. Define $\mathcal{E}_1 \coloneqq \ker(\mathrm{pr}_2 \circ \psi)$, which is a locally free sheaf on $\mathbb{P}^1$. By comparing determinants in the short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{O} \to 0$ we see that $\mathcal{E}_1$ is free, hence by an Ext^1 computation the sequence splits. The property $\mathrm{im}(\varphi) \subseteq \mathcal{E}_1$ follows from the definition. □

Proposition 5.11. *Let $f_1, f_2 \in |\mathcal{O}(d)|$ span the line $T \subseteq |\mathcal{O}(d)|$ and* $\mathrm{coker}(M|_T) \simeq \mathcal{O}^{\lambda_0} \oplus \bigoplus_{i=1}^s \mathcal{O}(d_i)$. *Define $U \coloneqq H^0(\mathbb{P}^n, \mathcal{O}(k-d))$. We have*

$$\lambda_0 = \dim H^0(\mathbb{P}^n, \mathcal{O}(k)) - \dim(f_1 U + f_2 U),$$

or, equivalently,

$$s = \dim(f_1 U + f_2 U) - d^{(k)}.$$

Proof. Note that the map $M|_T$ sends a local section $\xi \otimes \theta$ to $s\xi \otimes f_1\theta + t\xi \otimes f_2\theta$. In particular, the image of $\mathcal{O}(-1) \otimes U$ is contained in $\mathcal{O} \otimes (f_1 U + f_2 U)$. It follows that $\lambda_0 \geq \dim H^0(\mathbb{P}^n, \mathcal{O}(k)) - \dim(f_1 U + f_2 U)$.

To prove the other inequality, consider the induced sequence

$$0 \to \mathcal{O}(-1) \otimes U \xrightarrow{M|_T} \mathcal{O} \otimes (f_1 U + f_2 U) \to \mathcal{E}'' \to 0$$

and assume for a contradiction that $\mathcal{E}'' \simeq \mathcal{E}_1'' \oplus \mathcal{O}$. By Lemma 5.10, we have a splitting $\mathcal{O} \otimes (f_1 U + f_2 U) \simeq \mathcal{E}_1 \oplus \mathcal{O}$ such that $\mathrm{im}(M|_T) \subseteq \mathcal{E}_1$.

Consider the map $\widetilde{M}|_T \colon (\mathcal{O} \otimes U) \oplus (\mathcal{O} \otimes U) \to \mathcal{O} \otimes (f_1 U + f_2 U)$ defined by

$$\widetilde{M}|_T(a \otimes \theta_1, b \otimes \theta_2) = a \otimes f_1\theta_1 + b \otimes f_2\theta_2.$$

We obtain the matrix description of $\widetilde{M}|_T$ from the matrix description of $M|_T$ as follows. If $M|_T$ is represented by the matrix A with coefficients $A_{i,j} = \lambda_{i,j} s + \mu_{i,j} t$, then $\widetilde{M}|_T$ is represented by a block matrix

$$B = \left(\begin{array}{c|c} A' & A'' \end{array} \right)$$

with $A'_{i,j} = \lambda_{i,j}$ and $A''_{i,j} = \mu_{i,j}$.

The property $\mathrm{im}(M|_T) \subseteq \mathcal{E}_1$ implies that after some row operations, the matrix A has a zero row. By the construction of $\widetilde{M}|_T$, the same row operations lead to the matrix B having a zero row, but this is a contradiction, since the map $\widetilde{M}|_T$ is surjective. □

Corollary 5.12. *Let $t \in \mathrm{Gr}(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ be a line spanned by the polynomials f_1, f_2. Let k be such that $b^{(k)} = 0$, that is such that in the generic splitting type, only ones and zeroes appear, e.g. for $k \leq 2d$. Let θ range over a monomial basis of $H^0(\mathbb{P}^n, \mathcal{O}(k-d))$. The bundle $V_{k,t}$ has general type if and only if $\langle f_1\theta, f_2\theta \mid \theta \rangle$ is a linearly independent set in $H^0(\mathbb{P}^n, \mathcal{O}(k))$.*

Proof. Since $b^{(k)} = 0$, the type of $V_{k,t}$ is the generic splitting type if and only if it has $d^{(k)}$ many nonzero entries. By Proposition 5.11, this is the case if and only if $\dim \langle f_1\theta, f_2\theta \mid \theta\rangle = 2d^{(k)}$. ☐

Corollary 5.13. *Let $t \in \mathrm{Gr}(2, H^0(\mathbb{P}^n, \mathcal{O}(d)))$ be a line spanned by the polynomials f_1, f_2, and let k be such that $b^{(k)} = 0$. The bundle $V_{k,t}$ has non-generic splitting type if and only if $\deg(\gcd(f_1, f_2)) \geq 2d - k$. In particular, if $b^{(k)} = 0$ but $k > 2d$ then the generic type never occurs.*

Proof. By Corollary 5.12, the bundle $V_{k,t}$ has non-generic type if and only if there exist linearly independent $g_1, g_2 \in H^0(\mathbb{P}^n, \mathcal{O}(k-d))$ such that $g_1f_1 + g_2f_2 = 0$. Let $h := \gcd(f_1, f_2)$ and $d' := \deg h$.

If $d' \geq 2d - k$ then $\deg(f_i/h) \leq k - d$ and we may take g_1, g_2 to be multiples of f_1/h and f_2/h, respectively.

On the other hand, given such g_1 and g_2, we have $f_1 \mid g_2f_2$, which implies $f_1/h \mid g_2$, hence $d - d' \leq k - d$. ☐

Proposition 5.14. *Let $k = d + 1$. No types of V_k other than $(1, \dots, 1, 0, \dots, 0)$ and $(2, 1, \dots, 1, 0, \dots, 0)$ occur.*

Proof. Assume that the type of V_k at some line (f_1, f_2) is other than the two above. Then the type has at least two more zero entries than the general type. By Proposition 5.11, we have $\dim \langle f_1\theta, f_2\theta \mid \theta\rangle \leq 2d^{(k)} - 2$, so we find $g_1, g_2, g_1', g_2' \in H^0(\mathbb{P}^n, \mathcal{O}(1))$ and two linearly independent equations

$$\begin{aligned} g_1f_1 + g_2f_2 &= 0 \\ g_1'f_1 + g_2'f_2 &= 0, \end{aligned}$$

with both sets $(g_1, g_2), (g_1', g_2')$ linearly independent. From the first equation it follows that $f_1 = g_2h$ and $f_2 = -g_1h$, for some common factor h. Applying this to the second equation, we find $g_1'g_2 = g_2'g_1$, hence $g_1' = \alpha g_1$ and $g_2' = \alpha g_2$ for some scalar α, a contradiction. ☐

Example 5.15. Let $n = 3, d = 4$. Of the five type candidates

$$(1,1,1,1,0,\dots,0),\ (2,1,1,0,\dots,0),\ (2,2,0,\dots,0),\ (3,1,0,\dots,0),\ (4,0,\dots,0)$$

for V_5, only the first two occur as types of some $V_{5,t}$.

Proposition 5.16. *Let $k = 2d$. The most generic (i.e. smallest) splitting type of V_{2d} that is attained at some line is $(2, 1, \dots, 1, 0, \dots, 0)$.*

Proof. By Proposition 5.14, the type $(1,\dots,1,0,\dots,0)$ does not occur. As all other types are larger than $\sigma := (2,1,\dots,1,0,\dots,0)$, it suffices to prove that there exists a line where V_{2d} has type σ. Consider the line T spanned by $f_1 := x_0^d$ and $f_2 := x_1^d$. Letting θ range over the monomial basis of $H^0(\mathcal{O}(d))$, whe have $\dim \langle f_1\theta, f_2\theta \mid \theta\rangle = \dim H^0(\mathcal{O}(d)) - 1$ since the only nontrivial linear equation in the above set of vectors is $f_1 f_2 - f_2 f_1 = 0$. By Proposition 5.11, the type of $V_{2d}|_T$ has exactly one zero entry more than the non-occurring type $(1,\dots,1,0,\dots,0)$. But the only such type is σ. ☐

6 Loci of Types in the Grassmannian

In this section, we study the subsets of the grassmannian of lines in $|\mathcal{O}(d)|$ corresponding to the different splitting types for V_k. For this section, let $d \le k < 2d$, so that the generic splitting type is surely attained. We focus on the generic splitting type and the complement of its corresponding set in the Grassmannian, the set of jumping lines. As expected, the set corresponding to the generic splitting type is an open dense subset of the Grassmannian, and the other loci are locally closed. For $k = d+1$, and $n = 3$ we describe the cohomology class of the set of jumping lines in terms of Schubert cells.

6.1 Loci of types and the set of jumping lines

Definition 6.1. Let $k \ge 1$ and (b_i) be a splitting type for V_k. We define the set $Z_{(b_i)}$ of all points $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$ such that $V_{k,t}$ has splitting type (b_i). For the set of points t where $V_{k,t}$ has generic splitting type, we also write Z_{gen}, and define the *set of jumping lines* $Z := \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|) \setminus Z_{gen}$

Proposition 6.2. *The set Z_{gen} is Zariski open. Its complement Z is the union*

$$\mathrm{Supp}(R^1\varphi_* p^* V_k(-b^{(k)} - 1)) \cup \mathrm{Supp}(R^1\varphi_*(p^* V_k(-b^{(k)})^\vee)).$$

Proof. We begin by characterizing the set Z_{gen} via cohomology. Let $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$, write $V_{k,t} = \bigoplus_{i=1}^r \mathcal{O}(b_i)$ and $b := b^{(k)}$. We have $t \in Z_{\mathrm{gen}}$ if and only if $b \le b_i \le b+1$ for all i, which holds if and only if $H^1(P_t, V_{k,t}(-b-1)) = H^1(P_t, V_{k,t}(-b)^\vee) = 0$.

Next, we want to apply the Cohomology and Base Change Theorem [Vak17, 28.1.6] to the map $\varphi\colon P \to \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$, which is a $\mathbb{P}^1$-bundle, proper and flat. The last property ensures that locally free sheaves on P are flat over $\mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$.

For all $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$ we have $h^2(P_t, p^* V_{k,t}(-b-1)) = 0$ and $h^2(P_t, p^* V_{k,t}(-b)^\vee) = 0$. Since the sheaves $p^* V_{k,t}(-b-1)$ and $p^* V_{k,t}(-b)^\vee$ are locally free and coherent, we have

$$(R^1\varphi_* p^* V_k(-b-1))_t = H^1(P_t, V_{k,t}(-b-1))$$

and

$$(R^1\varphi_*(p^* V_k(-b)^\vee))_t = H^1(P_t, V_{k,t}(-b)^\vee).$$

By the previous characterization, we have

$$Z = \operatorname{Supp}(R^1\varphi_* p^* V_k(-b-1)) \cup \operatorname{Supp}(R^1\varphi_*(p^*V_k(-b)^\vee)),$$

which is a Zariski closed set. □

Proposition 6.3. *The sets* $\operatorname{Supp}(R^1\varphi_* p^* V_k(-b^{(k)}-1))$ *and* $\operatorname{Supp}(R^1\varphi_*(p^*V_k(-b^{(k)})^\vee))$ *are determinantal varieties in the sense of [ACG85, Ch. II, §4]*

Proof. To simplify notation, set $r_1 := \dim H^0(\mathbb{P}^n, \mathcal{O}(k)), r_2 := \dim H^0(\mathbb{P}^n, \mathcal{O}(k-d))$ and $b := b^{(k)}$, and rewrite the exact sequence from Proposition 4.4 as

$$0 \to \mathcal{O}(-1)^{r_2} \to \mathcal{O}^{r_1} \to V_k \to 0. \tag{6.1}$$

Twisting the sequence (6.1) with $\mathcal{O}(-b-1)$ and pulling back to P gives an exact sequence

$$0 \to p^*\mathcal{O}(-b-2)^{r_2} \to p^*\mathcal{O}(-b-1)^{r_1} \to p^*V_k(-b-1) \to 0.$$

For all $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$ we have $h^2(P_t, \mathcal{O}(-b-2)^{r_2}) = 0$, hence $R^2\varphi_* p^*\mathcal{O}(-b-2)^{r_2} = 0$ and applying φ_* to the above sequence gives an exact sequence

$$R^1\varphi_* p^*\mathcal{O}(-b-2)^{r_2} \xrightarrow{\alpha} R^1\varphi_* p^*\mathcal{O}(-b-1)^{r_1} \to R^1\varphi_* p^* V_k(-b-1) \to 0.$$

Note that since the numbers $h^1_2 := h^1(P_t, \mathcal{O}(-b-2)^{r_2})$ and $h^1_1 := h^1(P_t, \mathcal{O}(-b-1)^{r_1})$ do not depend on the point t, Grauert's Theorem applies, and the first two terms of the above sequence are locally free and coherent of rank h^2_1 and h^1_1, respectively. Since taking the fiber is right-exact, we see that for all t we have $(R^1\varphi_* p^* V_k(-b-1))_t \neq 0$ if and only if $\operatorname{coker}(\alpha_t) \neq 0$. Concluding, we have

$$\operatorname{Supp}(R^1\varphi_*(p^*V_k(-b-1))) = \{t : \operatorname{rk}(\alpha_t) \leq h^1_1 - 1\}.$$

As a final remark, note that $h^1_1 = br_1 = b\binom{k+n}{n}$.

The proof for the second assertion is similar. We start with the sequence (6.1), twist with $\mathcal{O}(-b)$, take duals, pull back to P, and apply φ_*. Since for all $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$ we have $h^1(P_t, \mathcal{O}(b)^{r_1}) = 0$, we obtain an exact sequence

$$\varphi_* p^*\mathcal{O}(b)^{r_1} \xrightarrow{\beta} \varphi_* p^*\mathcal{O}(b+1)^{r_2} \to R^1\varphi_*(p^*V_k(-b)^\vee) \to 0.$$

Since the numbers $h^0_1 := h^0(P_t, \mathcal{O}(b)^{r_1})$ and $h^0_2 := h^0(P_t, \mathcal{O}(b+1)^{r_2})$ do not depend on the point t, again by Grauert's Theorem the first two terms of the sequence are locally free of rank h^0_1 and h^0_2, respectively. As before, we obtain the characterization

$$\operatorname{Supp}(R^1\varphi_*(p^*V_k(-b)^\vee)) = \{t : \operatorname{rk}(\beta_t) \leq h^0_2 - 1\}.$$

Here, we have $h^0_2 = (b+2)r_2 = (b+2)\binom{k+n-d}{n}$. □

Proposition 6.4. *Let (b_i) be a type candidate for V_k. The set $\widehat{Z}_{(b_i)} \coloneqq \bigcup_{(b'_i)\geq(b_i)} Z_{(b'_i)}$ is Zariski-closed. In particular, the set $Z_{(b_i)}$ is locally closed.*

Proof. Let $t \in \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$ and $V_{k,t} = \bigoplus_{i=1}^{r^{(k)}} \mathcal{O}(b'_i)$. We have

$$\bigwedge^s V_{k,t} = \bigoplus_I \mathcal{O}(b'_I),$$

where I runs over the subsets of $\{1,\dots,r^{(k)}\}$ of size s and $b'_I \coloneqq \sum_{i\in I} b'_i$. For every type candidate (b'_i), the sum $\sum_{i=1}^s b'_i$ is the largest sum of s entries of (b'_i). Since $b'_i \geq 0$, the condition $\sum_{i=1}^s b'_i \geq \sum_{i=1}^s b_i$ is equivalent to the condition $h^0((\bigwedge^s V_{t,k})(-\sum^s b_i)) > 0$. Thus, we have

$$\widehat{Z}_{(b_i)} = \bigcap_{s=1}^{r^{(k)}} \{t : h^0((\textstyle\bigwedge^s V_{t,k})(-\sum^s b_i)) > 0\}.$$

With Serre duality and the Cohomology and Base Change theorem we write the sets of the intersection as

$$\mathrm{Supp}(R^1\varphi_*(p^*(\textstyle\bigwedge^s V_k^\vee)(\sum^s b_i - 2))),$$

which is Zariski-closed. □

Corollary 6.5. *Let (b_i) and (b'_i) be type candidates. If $Z_{(b_i)} \subseteq \overline{Z_{(b'_i)}}$ then $(b_i) \geq (b'_i)$.*

6.2 The components of the set of jumping lines

Definition 6.6. Let $d \leq k < 2d$ and $i = 0,\dots,k-d-1$. We define the subvariety $Q_i \subset |\mathcal{O}(d)|$ as the image of the map

$$f_i\colon |\mathcal{O}(k-d-i)| \times |\mathcal{O}(2d-k+i)| \to |\mathcal{O}(d)|$$

defined by $f_i(g,h) = gh$, and the map

$$\varphi_i\colon \mathbb{G}\mathrm{r}(1, |\mathcal{O}(k-d-i)|) \times |\mathcal{O}(2d-k+i)| \to \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|)$$

given by $\varphi_i((sg_1 + tg_2)_{(s:t)}, h) = (sg_1h + tg_2h)_{(s:t)}$.

Proposition 6.7. *Let $d \leq k < 2d$. The set of jumping lines Z is the union*

$$Z = \bigcup_{i=0}^{k-d-1} \operatorname{im}\varphi_i.$$

The subvarieties $\operatorname{im}\varphi_i$ have dimension $2(\binom{k-d+n-i}{n} - 2) + \binom{2d-k+n+i}{n}$. In particular for $k = 1$ the subvariety Z has codimension $2n + \binom{d+n-1}{n} + 3$.

Proof. The first statement follows from Corollary 5.13. For the statement about the dimension, note that the maps f_i are finite: the number of preimages of a point $q \in |\mathcal{O}(d)|$ is the number of ways to decompose q into a product gh, with $\deg(g) = k - d - i$ and $\deg(h) = 2d - k + 1$, up to scalars. It is in any case finite. By a similar argument we see that the maps φ_i are also finite, from which the statement about the dimensions follows. □

6.3 The expected codimension of Z

Let $k = d + 1$. The general type of V_k is $(1_{d^{(k)}}, 0_{r^{(k)}-d^{(k)}})$, and the nongeneric locus $Z \subseteq \mathbb{Gr}(1, |\mathcal{O}(d)|)$ is the determinantal variety $\operatorname{Supp}(R^1\varphi_* p^* V_k^\vee)$, the locus of singularity of the map

$$\varphi_* p^* \mathcal{O}^{\oplus d^{(k)}+r^{(k)}} \to \varphi_* p^* \mathcal{O}(1)^{\oplus d^{(k)}}$$

The ranks of the above bundles are $d^{(k)} + r^{(k)}$ and $2d^{(k)}$ respectively, so the expected codimension of Z_{gen} as a determinantal variety is $r^{(k)} - d^{(k)} + 1$ in this case. However, this is not the actual codimension, for example for $n = 3$ and $d = 4$ we have

$$\operatorname{codim} Z = 66 - (19 + 4) = 43 \neq 49 = r^{(4)} - d^{(4)} + 1.$$

Hence we cannot use the theorems about determinantal varieties of the expected codimension.

A similar problem arises when trying to consider the map $\mathbb{Gr}(1, |\mathcal{O}(d)|) \to \operatorname{Vect}_{\mathbb{P}^1}$ given by restricting V_{d+1} to lines. The codimension of the analogously defined locus $\{\underline{b} \in \operatorname{Vect}_{\mathbb{P}^1} : \underline{b} \geq (2, 1, \ldots, 1, 0, \ldots, 0)\}$ can be computed via the formula in [Lau90, §5]. For $n = 3$ and $d = 4$ this still gives a codimension of 49, so it seems for example that we do not have an immediate description of the cohomology class of Z as the pullback of some class in $\operatorname{Vect}_{\mathbb{P}^1}$.

6.4 The cohomology class of the set of jumping lines

To perform calculations in the Chow ring A of $\mathbb{Gr}(1, |\mathcal{O}(d)|)$, we follow the conventions found in [EH16]. We assume $\operatorname{char}(k) = 0$ for simplicity. Let $N := \dim H^0(\mathcal{O}(d)) = \binom{n+d}{n}$. For $N - 2 \geq a \geq b$, we have the Schubert cycle

$$\Sigma_{a,b} := \{T \in \mathbb{Gr}(1, |\mathcal{O}(d)|) : T \cap H \neq \varnothing, T \subseteq H'\},$$

where $(H \subset H')$ is a general flag of linear subspaces of dimension $N-a-2$ resp. $N-b-1$ in the projective space $|\mathcal{O}(d)|$. The ring A is generated by the Schubert classes $\sigma_{a,b}$ of the cycles $\Sigma_{a,b}$. The class $\Sigma_{a,b}$ has codimension $a+b$, and we use the convention $\sigma_a := \sigma_{a,0}$.

We calculate the cohomology class of the set of jumping lines Z of the Verlinde bundle $V_{d+1,t}$ of the family of hypersurfaces of degree $d + 1$ in $\mathbb{P}^n$, with $n \leq 3$.

Proposition 6.8. *Let $n \leq 3$ and let Z be set of jumping lines of $V_{d+1,t}$, and let $[Z]$ be the class of Z in the Chow ring A. Let b range over the integers with the property $0 \leq b < \frac{\dim Z}{2}$ and define $a = \dim Z - b, a' = a + \lfloor \frac{\operatorname{codim} Z}{2} \rfloor$, $b' = b + \lfloor \frac{\operatorname{codim} Z}{2} \rfloor$.*

(i) *If $\dim Z$ is odd or $n = 2$, we have*

$$[Z] = \sum_{a,b} \left(\binom{a+1}{n}\binom{b+1}{n} - \binom{a+2}{n}\binom{b}{n} \right) \sigma_{a',b'}. \tag{6.2}$$

(ii) *If $\dim Z$ is even and $n = 3$, we have*

$$[Z] = \sum_{a,b} \left(\binom{a+1}{n}\binom{b+1}{n} - \binom{a+2}{n}\binom{b}{n} \right) \sigma_{a',b'} + \binom{\frac{\dim Z}{2}+2}{n}\binom{\frac{\dim Z}{2}}{n} \sigma_{\frac{\dim Z}{2},\frac{\dim Z}{2}}.$$

Proof. Let $Q \subset |\mathcal{O}(d)|$ be the image of the multiplication map

$$f\colon |\mathcal{O}(1)| \times |\mathcal{O}(d-1)| \to |\mathcal{O}(d)|$$

as in Definition 6.6. The map f is birational on its image, since a general point of Q has the form gh with h irreducible. By Proposition 6.7, the variety Z is the image of the finite multiplication map

$$\varphi\colon \mathbb{G}\mathrm{r}(1, |\mathcal{O}(1)|) \times |\mathcal{O}(d-1)| \to \mathbb{G}\mathrm{r}(1, |\mathcal{O}(d)|).$$

The Chow group $A^{\operatorname{codim} Z}$ is generated by the classes $\sigma_{a',b'}$ with $N-2 \geq a' \geq b' \geq \lfloor \frac{\operatorname{codim} Z}{2} \rfloor$ and $a' + b' = \operatorname{codim} Z$, while the complementary group $A^{\dim Z}$ is generated by the classes $\sigma_{\dim Z - b, b}$ with $b \in 0, \ldots, \lfloor \frac{\dim Z}{2} \rfloor$. Write

$$[Z] = \sum_{a',b'} \alpha_{a',b'} \sigma_{a',b'}.$$

We have $\sigma_{a',b'}\sigma_{a,b} = 1$ if $b' - b = \lfloor \frac{\operatorname{codim} Z}{2} \rfloor$ and 0 else. Hence, multiplying the above equation with the complementary classes $\sigma_{a,b}$ and taking degrees gives $\alpha_{a',b'} = \deg([Z] \cdot \sigma_{a,b})$.

Using Giambelli's formula $\sigma_{a,b} = \sigma_a\sigma_b - \sigma_{a+1}\sigma_{b-1}$ [EH16, Prop. 4.16], we reduce to computing $\deg([Z] \cdot \sigma_a\sigma_b)$ for $0 \leq b \leq \lfloor \frac{\dim Z}{2} \rfloor$. By Kleiman transversality, we have

$$\deg([Z] \cdot \sigma_a\sigma_b) = |\{T \in Z : T \cap H \neq \varnothing, T \cap H' \neq \varnothing\}|,$$

where H and H' are general linear subspaces of $|\mathcal{O}(d)|$ of dimension $N - a - 2$ and $N - b - 2$, respectively.

To a point $p = g_p h_p \in Q$ with $g_p \in |\mathcal{O}(1)|$ and $h_p \in |\mathcal{O}(d-1)|$, associate a closed reduced subscheme $\Lambda_p \subset Q$ containing p as follows. If h_p is irreducible, let Λ_p be the image of the linear embedding $|\mathcal{O}(1)| \times \{h_p\} \to |\mathcal{O}(d)|$ given by $g \mapsto gh_p$.

If h_p is reducible, define the subscheme Λ_p as the union $\bigcup_h \operatorname{im}(|\mathcal{O}(1)| \times \{h\} \to |\mathcal{O}(d)|)$, where h ranges over the (up to multiplication by units) finitely many divisors of p of degree $d-1$.

Note that for all points p, the spaces $\operatorname{im}(|\mathcal{O}(1)| \times \{h\} \to |\mathcal{O}(d)|)$ meet exactly at p.

By the definition of Z, all lines $T \in Z$ lie in Q. Furthermore, if T meets the point p, then $T \subseteq \Lambda_p$. For $H \subseteq |\mathcal{O}(d)|$ a linear subspace of dimension $N-a-2$, define $Q' := H \cap Q$. For general H, the subscheme Q' is a smooth subvariety of dimension $b-n+1$ such that for a general point $p = gh$ of Q' with $h \in |\mathcal{O}(d)|$, the polynomial h is irreducible.

Next, we consider the case $n = 2$ or $\dim Z$ odd. We show that for general H, for each point $p \in Q'$ we have $\Lambda_p \cap H = \{p\}$. Let $\mathcal{H}$ denote the parameter space for H, i.e. the Grassmannian $\operatorname{Gr}(\dim H + 1, N)$. Define the closed subset $X \subseteq Q \times \mathcal{H}$ by $X := \{(p, H) : \dim(H \cap \Lambda_p) \geq 1\}$. The fibers of the induced map $X \to \mathcal{H}$ have dimension at least one. Hence, to prove that the desired condition on H is an open condition, it suffices to prove $\dim(X) \leq \dim(\mathcal{H})$. The fiber of the map $X \to Q$ over a point p consists of the union of finitely many closed subsets of the form $X'_p = \{H \in \mathcal{H} : \dim(H \cap \Lambda'_p) \geq 1\}$, where $\Lambda'_p \simeq \mathbb{P}^n \subseteq |\mathcal{O}(d)|$ is one of the components of Λ_p. The space X'_p is a Schubert cycle

$$\Sigma_{\dim Q-b, \dim Q-b} = \{H \in \operatorname{Gr}(\dim H + 1, N) : \dim(H \cap H_{n+1}) \geq 2\},$$

with H_{n+1} an $(n+1)$-dimensional subspace of $H^0(\mathcal{O}(d))$. The codimension of the cycle is $2(\dim Q - b)$, hence also $\operatorname{codim}(X_p) = 2(\dim Q - b)$. Finally, we have $\dim(\mathcal{H}) - \dim(X) = \operatorname{codim}(X_p) - \dim(Q) = \dim Q - 2b$. If $\dim Z$ is odd, then $\dim Q - 2b \geq \dim Q - \dim Z + 1 = 3 - n \geq 0$. If $n = 2$, we instead estimate $\dim Q - 2b \geq \dim Q - \dim Z = 2 - n \geq 0$.

Next, let $\Lambda := \bigcup_{p \in Q'} \Lambda_p = f(|\mathcal{O}(1)| \times \operatorname{pr}_2 f^{-1}(Q'))$ and $\Lambda'' := |\mathcal{O}(1)| \times \operatorname{pr}_2 f^{-1}(Q')$. By the choice of H, the map $f^{-1}(Q') \to Q'$ is birational and the map $f^{-1}(Q') \to \operatorname{pr}_2 f^{-1}(Q')$ is even bijective. It follows that Λ'' and hence Λ have dimension $b+1$. The intersection of Λ with a general linear subspace H' of dimension $N-b-2$ is a finite set of points. For each point $p \in Q'$, the linear subspace H' intersects each component Λ'_p of Λ_p in at most one point. For each point $p' \in H' \cap \Lambda$ there exists a unique p such that $p' \in \Lambda_p$. Furthermore, the only line $T \in Z$ meeting both p and H' is the one through p and p'. If the intersection $H' \cap \Lambda_p$ is empty, then there will be no line meeting p and H'. Hence, $\deg([Z] \cdot \sigma_a \sigma_b)$ is the number of intersection points of Λ with a general H'.

Finally, the pre-image $f^{-1}(Q') = f^{-1}(H)$ is smooth for a general H by Bertini's Theorem. If ζ is the class of a hyperplane section of $|\mathcal{O}(d)|$ we have $f^*(\zeta) = \alpha + \beta$, , where α and β are classes of hyperplane sections of $|\mathcal{O}(1)|$ and $|\mathcal{O}(d)|$, respectively. Since pr_2 and f

have degree one, we compute:

$$\begin{aligned}
[\Lambda''] &= [\mathrm{pr}_2^{-1}\mathrm{pr}_2 f^{-1}(H)] \\
&= \mathrm{pr}_2^*[\mathrm{pr}_2 f^{-1}(H)] \\
&= \mathrm{pr}_2^*\mathrm{pr}_{2,*}[f^{-1}(H)] \\
&= \mathrm{pr}_2^*\mathrm{pr}_{2,*}f^*[H] \\
&= \mathrm{pr}_2^*\mathrm{pr}_{2,*}(\alpha+\beta)^{\operatorname{codim} H} \\
&= \tbinom{\operatorname{codim} H}{n}\mathrm{pr}_2^*\beta^{\operatorname{codim} H-n} \\
&= \tbinom{\operatorname{codim} H}{n}\beta^{\operatorname{codim} H-n}
\end{aligned}$$

Hence, by the push-pull formula:

$$\begin{aligned}
\deg([\Lambda]\cdot H') &= \deg([\Lambda'']\cdot(\alpha+\beta)^{\operatorname{codim} H'}) \\
&= \binom{\operatorname{codim} H}{n}\binom{\operatorname{codim} H'}{n} \\
&= \binom{a+1}{n}\binom{b+1}{n}.
\end{aligned}$$

We then use Giambelli's formula to obtain Equation (6.2).

In case $n=3$ and $\dim Z$ even, we need to show that for $b=\dim Z/2$ we have $\deg([Z]\cdot\sigma_{b,b})=0$. In this case, the hyperplanes H and H' have the same dimension $N-b-2$.

For $p\in Q$, the set Λ_p is defined as before. We now claim that for general H of dimension $N-b-2$, we have $\dim(\Lambda_p\cap H)=1$. Consider as before the closed subscheme $X:=\{(p,H):\dim(\Lambda_p\cap H)\geq 1\}\subset Q\times\mathcal{H}$. The generic fiber of the projection map $\varphi\colon X\to\mathcal{H}$ is one-dimensional, hence we have $\dim\varphi(X)=\dim(X)-1=\dim\mathcal{H}$. The last equation holds with $n=3$ and $2b=\dim Z$. Hence for all $H\in\mathcal{H}$ we have $\dim(\Lambda_p\cap H)\geq 1$.

On the other hand, the equality $\dim(\Lambda_p\cap H)=1$ is attained by some, and hence by a general, H. Indeed, consider the closed subscheme $\widetilde{X}:=\{(p,H):\dim(\Lambda_p\cap H)\geq 2\}\subset Q\times\mathcal{H}$. By a similar argument as before, one needs to show that $\dim(\mathcal{H})-\dim(X)+1\geq 0$. The fiber X_p is a Schubert cycle of codimension $3(\dim Q-b+1)$. Lastly, a computation shows $\dim(\mathcal{H})-\dim(\widetilde{X})+1=\operatorname{codim}(\widetilde{X}_p)-\dim(Q)+1=\frac{1}{2}(2\dim Q+18-5n)\geq 0$.

Now, define Λ'' as above. We have $\dim\Lambda''=\dim|\mathcal{O}(1)|+\dim\mathrm{pr}_2 f^{-1}(Q')=b$. Since f is generically of degree one, we still have $\dim\Lambda''=\Lambda$, hence $\dim\Lambda+\dim H'=N-2<\dim|\mathcal{O}(d)|$. It follows that a generic H' does not meet any of the lines $T\subset Z$, hence $\sigma_b\sigma_b\cdot[Z]=0$.

□

7 Global Properties of the Verlinde Bundles

In this final section, we study global properties of V_k. We would like for example to know if V_k is stable. A vector bundle V on projective space is *stable* if $\mu(V') < \mu(V)$ for every subbundle $V' \subsetneq V$. Here, $\mu(V) \coloneqq \frac{\deg(V)}{\operatorname{rk} V}$ is the *slope* of V. Even though the question of stability remains open, we point to some evidence that V_k is stable. We conclude by showing that for $d = 4$, the bundle V_5 is *irreducible*, i.e. not decomposable as a nontrivial direct sum of vector bundles.

7.1 Subbundles with prescribed splitting type

Proposition 7.1. *There exists no subbundle $\mathcal{O}(1) \subset V_k$.*

Proof. By twisting the sequence (4.1) with $\mathcal{O}(-1)$ and taking cohomology we obtain $H^0(V_k(-1)) = 0$, hence there are no subbundles $\mathcal{O} \subset V_k(-1)$. □

Corollary 7.2. *There exists no subbundle $W \subset V_k$ such that the splitting type of $W|_T$ is $\mathcal{O}(1)^{\oplus \operatorname{rk} W}$ for all W.*

Proof. By [OSS80, Thm. 3.2.1], a vector bundle of trivial splitting type for all lines through a point is trivial, hence $W|_T$ would be trivial, in contradiction to Proposition 7.1. □

7.2 Projective flatness

To consider questions about projective flatness, let $k = \mathbb{C}$. A vector bundle V of rank r on a complex manifold X is *projectively flat* if its projectivization $\mathbb{P}(V)$ is isomorphic as a principal $\mathrm{PGL}(r, \mathbb{C})$-bundle to the bundle P_ρ associated to a representation $\rho\colon \pi_1(X) \to \mathrm{PGL}(r, \mathbb{C})$. The bundle P_ρ is defined as $P_\rho \coloneqq \widetilde{X} \times_{\pi_1(X)} \mathrm{PGL}(r, \mathbb{C})$, where $\widetilde{X}$ denotes the universal covering space of X.

Proposition 7.3. *Let V be a vector bundle on $|\mathcal{O}(d)|$. If the restriction $V|_{\mathrm{sm}}$ of V to the smooth locus $|\mathcal{O}(d)|_{\mathrm{sm}}$ is projectively flat, then it is trivial.*

Proof. Let $r \coloneqq \operatorname{rk} V$, let T be a general line in $|\mathcal{O}(d)|$, and let $T_{\mathrm{sm}} \coloneqq T \cap |\mathcal{O}(d)|_{\mathrm{sm}}$. Assuming V_{sm} is projectively flat, its projectivisation is given by a representation

$$\rho\colon \pi_1(|\mathcal{O}(d)|_{\mathrm{sm}}) \to \mathrm{PGL}(r, \mathbb{C}).$$

Then $V_{\mathrm{sm}}|_{T_{\mathrm{sm}}}$ is also projectively flat, its projectivisation given by a representation $\rho' \colon \pi_1(T|_{\mathrm{sm}}) \to \mathrm{PGL}(r, \mathbb{C})$ fitting into a commutative diagram

$$\begin{array}{ccc} \pi_1(|\mathcal{O}(d)|_{\mathrm{sm}}) & \xrightarrow{\rho} & \mathrm{PGL}(r,\mathbb{C}) \\ {\scriptstyle\alpha}\uparrow & & \| \\ \pi_1(T_{\mathrm{sm}}) & \xrightarrow{\rho'} & \mathrm{PGL}(r,\mathbb{C}), \end{array}$$

where the map α is induced by the inclusion. From the non-singular, quasi-projective version of the Lefschetz hyperplane theorem proven in [HT85], it follows that the map α is surjective. Since $V_{\mathrm{sm}}|_{T_{\mathrm{sm}}}$ is the pullback of a vector bundle over $\mathbb{C}$, it is trivial, so ρ' is trivial. Hence ρ is trivial. □

Corollary 7.4. *The restriction $V_{k,\mathrm{sm}}$ of V_k to the locus of smooth hypersurfaces is not projectively flat. Furthermore, the restriction of V_k to the locus of semistable hypersurfaces is not projectively flat.*

Proof. The second statements follows from the first, since otherwise $V_{k,\mathrm{sm}}$ would be projectively flat. □

7.3 Stability

Conjecture 7.5. *The Verlinde bundles V_k are stable for all n, k, d.*

We will now try to see ways in which this conjecture is not trivially false. For example, the next statement is necessary for stable bundles.

Proposition 7.6. *Let H be the class of a hyperplane in $\mathrm{CH}^1(|\mathcal{O}(d)|)$, let $N = \dim|\mathcal{O}(d)|$. We have*

$$\int \mathrm{ch}_2(\mathrm{End}(V_k))H^{N-2} < 0.$$

Proof. With the sequence (4.1), one computes $\mathrm{ch}_1(V_k) = d^{(k)}H$ and $\mathrm{ch}_2(V_k) = -\frac{1}{2}d^{(k)}H^2$. With these equalities and $\mathrm{ch}_i(V_k) = (-1)^i \mathrm{ch}_i(V_k^\vee)$, we get

$$\begin{aligned} \mathrm{ch}_2(\mathrm{End}(V_k)) &= \mathrm{ch}_0(V_k)\,\mathrm{ch}_2(V_k^\vee) + \mathrm{ch}_1(V_k)\,\mathrm{ch}_1(V_k^\vee) + \mathrm{ch}_0(V_k^\vee)\,\mathrm{ch}_2(V_k) \\ &= -(r^{(k)}d^{(k)} + (d^{(k)})^2)H^2, \end{aligned}$$

hence $\mathrm{ch}_2(\mathrm{End}(V_k))H^{N-2}$ has negative degree. □

For $k' < k$, there are inclusions $\mathcal{O}\boxtimes\mathcal{O}(k') \hookrightarrow \mathcal{O}\boxtimes\mathcal{O}(k)$ on $|\mathcal{O}(d)|\times\mathbb{P}^n$ inducing inclusions $V_{k'} \subset V_k$. The next proposition show that these are not destabilizing.

Proposition 7.7. *Let $k' < k$. We have $\mu(V_{k'}) < \mu(V_k)$.*

Proof. It suffices to prove the statement for $k' = k - 1$. We compute

$$\begin{aligned}(\mu(V_{k'})^{-1} + 1)(\mu(V_k)^{-1} + 1)^{-1} &= \frac{\binom{n+k'}{n}\binom{n+k-d}{n}}{\binom{n+k'-d}{n}\binom{n+k}{n}} \\ &= \frac{k(n+k-d)}{(k+n)(k-d)} \\ &= \left(1 + \frac{n}{k}\right)^{-1}\left(1 + \frac{n}{k-d}\right) \\ &> 1,\end{aligned}$$

which shows that $\mu(V_{k-1}) < \mu(V_k)$. □

Stable bundles V are simple, i.e. they have $H^0(\operatorname{End} V) = \mathbb{C}$. In the case $\operatorname{rk} V > 1$, this would not be the case if $H^0(V), H^0(V^\vee) \neq 0$. The following proposition rules this out for V_k.

Proposition 7.8. *We have $H^0(V_k^\vee) = 0$.*

Proof. Let $M^\vee$ be the dual of M in the sequence (4.1). The map on global sections

$$H^0(M^\vee)\colon H^0(\mathcal{O}(k)) \to H^0(\mathcal{O}(d)) \otimes H^0(\mathcal{O}(k-d))$$

sends a section $\sum_{I_k} \lambda_{I_k} x^{I_k}$ to $\sum_{I_k} \sum_{I_d < I_k} \lambda_{I_k} x_{I_d} \otimes \frac{x^{I_k}}{x^{I_d}}$. The coefficient of $x_{I_{k-d}}$ in this expression is $\sum_{I_d} x_{I_d} \lambda_{(I_d + I_{k+d})}$, so we see that $H^0(M^\vee)$ is injective. □

7.4 Irreducibility

A reducible bundle is not stable, so it could be helpful, while interesting in its own right, to ask whether V_k is irreducible. We give an affirmative answer for $d = 4, n = 3, k = 5$.

Lemma 7.9. *Let V be a vector bundle on a scheme X with a decomposition $V = V_1 \oplus V_2$. Assume V fits into an exact sequence of vector bundles*

$$0 \to K \to H^0(V) \otimes \mathcal{O} \xrightarrow{\varphi} V \to 0, \tag{7.1}$$

where φ is the canonical evaluation map. Then there are exact sequences

$$\begin{aligned} 0 \to K_1 \to H^0(V_1) \otimes \mathcal{O} \to V_1 \to 0 \\ 0 \to K_2 \to H^0(V_2) \otimes \mathcal{O} \to V_2 \to 0, \end{aligned}$$

whose direct sum is the sequence (7.1).

Proof. The canonical map $H^0(V_1) \otimes \mathcal{O} \to V$ coming from the inclusion $V_1 \subset V$ has image contained in V_1 and is surjective, the same holds for V_2. One also verifies that $K_1 \oplus K_2 = K$. □

Remark 7.10. Taking cohomology of the sequence (4.1) shows that

$$H^0(V_k) \simeq H^0(\mathbb{P}^n, \mathcal{O}(k))$$

and that the composition $H^0(V_k) \otimes \mathcal{O} \to H^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O} \to V$ is just the canonical evaluation map.

Construction 7.11. To the map $M \colon \mathcal{O}(-1) \otimes H^0(\mathcal{O}(k-d)) \to \mathcal{O} \otimes H^0(\mathcal{O}(k))$ from the sequence (4.1) we associate the map $\widetilde{M} \colon H^0(\mathcal{O}(d)) \otimes H^0(\mathcal{O}(k-d)) \to H^0(\mathcal{O}(k))$ given by multiplication. Let M be given by the entries $(\sum_{I_d} \lambda_{I_d,i,j} x^{I_d})_{ij}$. The matrix $\widetilde{M}$ then looks as follows:

$$\widetilde{M} = \left(\begin{array}{c|c|c|c} A_{I_d^{(1)}} & A_{I_d^{(2)}} & \cdots & A_{I_d^{(N)}} \end{array} \right)$$

where for every index I_d, the matrix A_{I_d} is a matrix of the size of M with $(A_{I_d})_{i,j} = \lambda_{I_d,i,j}$. We note the following properties:

(i) *If M is a block matrix of the form $\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$, then all the A_{I_d} also are, and thus the matrix $\widetilde{M}$ can be brought in the same block form after suitably permuting its columns.*

(ii) *Row operations on M correspond to row operations on $\widetilde{M}$. One column operation on M corresponds to column operations on $\widetilde{M}$ performed on each of the A_{I_d}.*

(iii) *The map $\widetilde{M}$ is surjective.*

Proposition 7.12. *There exists no section $\mathcal{O} \hookrightarrow V_k$ that splits as a direct summand.*

Proof. Such a splitting would imply that one can perform row operations on the matrix M until it has a zero row. Hence the matrix $\widetilde{M}$ would also have a zero row. But this is impossible since $\widetilde{M}$ is surjective. □

Proposition 7.13. *There exists no direct summand V' of V_k with $c_1(V') \leq 1$.*

Proof. Note that every direct summand $V' \subset V_k$ is globally generated. If $c_1(V') = 0$, then V' is trivial by [OSS80, Thm. 3.2.1]. If $c_1(V') = 1$, then V' is uniform of splitting type $(1, 0, \ldots, 0)$. By [Ell82, IV – 2.2.: Prop], V' is either isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}^{\operatorname{rk} V' - 1}$ or to $T(-1) \oplus \mathcal{O}^{\operatorname{rk} V' - N - 1}$. Both cases contradict Proposition 7.12. □

Proposition 7.14. *Let $n = 3, d = 4$. The vector bundle V_5 is indecomposable.*

Proof. Since $c_1(V_5) = 4$, it suffices to prove that there exists no direct summand $V' \subset V_k$ with $c_1(V') = 2$. Let V' be such a direct summand, V'' its direct complement. By Lemma 7.9, the matrix M splits into a direct sum $M = M_1 \oplus M_2$ with $M_i \colon \mathcal{O}(-1)^{\oplus 2} \to H^0(V_i) \otimes \mathcal{O}$. Consider the corresponding splitting $\widetilde{M} = \widetilde{M_1} \oplus \widetilde{M_2}$. We have $\widetilde{M_i} \colon H^0(\mathcal{O}(5)) \otimes \langle f_i, g_i \rangle \to H^0(V_i)$ for some $f_i, g_i \in H^0(\mathcal{O}(5-4))$. Since the $\widetilde{M_i}$ are given by multiplication, we have $\operatorname{rk} \widetilde{M_i} \geq N + 1 = 35$. But then

$$70 = \operatorname{rk} M_1 + \operatorname{rk} M_2 \leq \dim H^0(\mathcal{O}(5)) = 56,$$

a contradiction. □

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