Jumping Lines of Verlinde Bundles of Families of Hypersurfaces

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1. Example

Consider the projective plane $\mathbb{P}^2$, with coordinates $(x_0 : x_1 : x_2)$.

On $\mathbb{P}^2$, the general quadric is $f = \alpha_{00}x_0^2 + \alpha_{01}x_0x_1 + \alpha_{02}x_0x_2 + \alpha_{11}x_1^2 + \alpha_{12}x_1x_2 + \alpha_{22}x_2^2$.

The space of quadrics on $\mathbb{P}^2$ is $|\mathcal{O}(2)| = \mathbb{P}^5$, with points

$$[f] = (\alpha_{00} : \alpha_{01} : \alpha_{02} : \alpha_{11} : \alpha_{12} : \alpha_{22}).$$

For each f, consider the map $M_f = (\cdot f) \colon H^0(\mathbb{P}^2, \mathcal{O}(1)) \to H^0(\mathbb{P}^2, \mathcal{O}(3))$

This gives a global map M represented by a matrix as below.

A line $T \subset |\mathcal{O}(2)|$ is given by $(sf + tg)_{(s:t)\in\mathbb{P}^1}$.

Let $f = \sum \lambda_I x^I$, $g = \sum \mu_I x^I$. One obtains M_T by substituting $\alpha_{ij} \leftarrow \lambda_{ij}s + \mu_{ij}t$.

Question: What does M_T reduce to after row and column operations?

Example:

First: $f = x_0^2 + x_1^2 + x_2^2, \quad g = x_0x_1 - x_1x_2$.

Second: $f = x_0^2 + 2x_0x_1 + x_1^2, \quad g = x_0x_2 - x_0x_1 + x_1x_2 - x_1^2$.

$$M = \begin{pmatrix} \alpha_{00} & & \\ \alpha_{01} & \alpha_{00} & \\ \alpha_{02} & & \alpha_{00} \\ \alpha_{11} & \alpha_{01} & \\ \alpha_{12} & \alpha_{02} & \alpha_{01} \\ \alpha_{22} & & \alpha_{02} \\ & \alpha_{11} & \\ & \alpha_{12} & \alpha_{11} \\ & \alpha_{22} & \alpha_{12} \\ & & \alpha_{22} \end{pmatrix} \to \begin{pmatrix} s & & \\ t & s & \\ & & s \\ s & t & \\ -t & & t \\ s & & \\ & s & \\ & -t & s \\ & s & -t \\ & & s \end{pmatrix} \to \begin{pmatrix} s & & \\ t & & \\ & & \\ & t & \\ -t & & t \\ & & \\ & s & \\ & -t & \\ & & -t \\ & & s \end{pmatrix} \to \begin{pmatrix} s & & \\ t & & \\ & s & \\ & t & \\ & & s \\ & & t \\ & & \\ & & \\ & & \\ & & \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{00} & & \\ \alpha_{01} & \alpha_{00} & \\ \alpha_{02} & & \alpha_{00} \\ \alpha_{11} & \alpha_{01} & \\ \alpha_{12} & \alpha_{02} & \alpha_{01} \\ \alpha_{22} & & \alpha_{02} \\ & \alpha_{11} & \\ & \alpha_{12} & \alpha_{11} \\ & \alpha_{22} & \alpha_{12} \\ & & \alpha_{22} \end{pmatrix} \to \begin{pmatrix} s & & \\ 2s-t & s & \\ t & & s \\ s-t & 2s-t & \\ t & t & 2s-t \\ & & t \\ & s-t & \\ & t & s-t \\ & & t \\ & & \end{pmatrix} \to \begin{pmatrix} s & & \\ -t & s & \\ t & & s \\ -t & s & \\ t & t & 2s \\ & & \\ t & -t & \\ & t & s \\ & & t \\ & & \end{pmatrix} \to \begin{pmatrix} s & & \\ t & s & \\ & t & \\ & & s \\ & & t \\ & & \\ & & \\ & & \\ & & \\ & & \end{pmatrix}$$

A priori, we expect[1] one of the types

$$(1_3, 0_4) = \begin{pmatrix} s & & \\ t & & \\ & s & \\ & t & \\ & & s \\ & & t \end{pmatrix}, (2_1, 1_1, 0_5) = \begin{pmatrix} s & & \\ t & s & \\ & t & \\ & & s \\ & & t \end{pmatrix} \text{ or } (3_1, 0_7) = \begin{pmatrix} s & & \\ t & s & \\ & t & s \\ & & t \end{pmatrix}.$$

corresponding to the tuples with 10 (rows of M) nonnegative entries that sum up to 3 (columns of M). We have the following

Theorem. *Let $T = (sg + tf)_{(s:t)}$ be a line of quadrics. If there exists a linear form h with $h \mid f, g$, then M_T has the type $(2_1, 1_1, 0_5)$. Otherwise, it has the type $(1_3, 0_4)$.*

For instance, in the second example above: $f = (x_0+x_1)^2$ and $g = (x_0+x_1)(x_2-x_1)$.

The Grassmannian $\mathbb{Gr}(1, |\mathcal{O}(2)|) \subset \mathbb{P}^{14}$ parametrizes the lines T.

The type $(2_1, 1_1, 0_5)$ occurs on the irreducible closed subvariety

$$Z \coloneqq \operatorname{im}(\mathbb{Gr}(1, |\mathcal{O}(1)|) \times |\mathcal{O}(1)| \to \mathbb{Gr}(1, |\mathcal{O}(2)|)),$$

the map being given by multiplication.

We have $\dim \mathbb{Gr}(1, |\mathcal{O}(2)|) = 8$ and $\dim Z = 4$. The following theorem allows one to compute the cohomology class of $[Z]$ in the Chow ring of the Grassmannian.

Theorem. *Let $a + b = 4$ and $a \geq b \geq 0$. Let $\Sigma_{a,b} \subset \mathbb{Gr}(1, |\mathcal{O}(2)|)$ denote the closed subset $\{T \mid T \cap H \neq 0, T \subseteq H'\}$, with $(H \subseteq H' \subseteq |\mathcal{O}(2)|)$ a general flag of linear subspaces of dimensions $4 - a$ and $3 - b$ respectively. Then*

$$|Z \cap \Sigma_{a,b}| = \binom{a+1}{2}\binom{b+1}{2} - \binom{a+2}{2}\binom{b}{2}.$$

[1] This is a bit imprecise

2. General Setup

Let $\pi\colon X \to S$ be a flat proper morphism of schemes, $\mathcal{L}$ an ample line bundle on X. To better understand the family π, one can study the pushforwards $V_k \coloneqq \pi_*(\mathcal{L}^{\otimes k})$, for $k \geq 1$. In good situations, V_k is a vector bundle on S and we have $V_k|_s = H^0(X_s, \mathcal{L}^{\otimes k})$ for $s \in S$. The V_k are then called the *Verlinde bundles* of the family π.

The following setup is treated in [Hem15] as an example of Verlinde Bundles.

Definition. Let $n > 1$ and $\pi\colon \mathfrak{X} \to |\mathcal{O}_{\mathbb{P}^n}(d)|$ be the universal family of hypersurfaces of degree d in $\mathbb{P}^n$. Let $\mathcal{L}$ be the restriction to $\mathfrak{X}$ of the bundle $\mathcal{O}(1) \boxtimes \mathcal{O}$ under the inclusion $\mathfrak{X} \subseteq \mathbb{P}^n \times |\mathcal{O}(d)|$.

The sheaf $\pi_*\mathcal{L}^{\otimes k}$ is locally free of rank $\binom{k+n}{n} - \binom{k+n-d}{n}$.

Let $k \geq 1$. The k-th *Verlinde bundle* of the family π is the vector bundle $V_k \coloneqq \pi_*\mathcal{L}^{\otimes k}$.

There exists a short exact sequence of vector bundles on $|\mathcal{O}(d)|$

$$0 \to \mathcal{O}(-1) \otimes H^0(\mathbb{P}^n, \mathcal{O}(k-d)) \xrightarrow{M} \mathcal{O} \otimes H^0(\mathbb{P}^n, \mathcal{O}(k)) \to V_k \to 0. \tag{1}$$

The map M is multiplication by $\sum_I \alpha_I \otimes x^I$ in $H^0(|\mathcal{O}(d)|, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(d))$.

3. Splitting Types

Definition. Let $T \subseteq |\mathcal{O}(d)|$ be a line. On $T = \mathbb{P}^1$, we define the vector bundle $V_{k,T} \coloneqq V_k|_T$. The *splitting type* of $V_{k,T}$ is the unique non-increasing tuple $(b_1, \ldots, b_{r(k)})$ such that $V_{k,T} \simeq \bigoplus_i \mathcal{O}(b_i)$.

By (1), the vector bundle $V_{k,T}$ has degree $\binom{k+n-d}{n}$ and rank $\binom{k+n}{n} - \binom{k+n-d}{n}$. Furthermore, all the b_i are nonnegative.

Theorem. *Let $f_1, f_2 \in |\mathcal{O}(d)|$ span the line $T \subseteq |\mathcal{O}(d)|$ and $\operatorname{coker}(M|_T) \simeq \mathcal{O}^{\lambda_0} \oplus \bigoplus_{i=1}^s \mathcal{O}(d_i)$. Define $U \coloneqq H^0(\mathbb{P}^n, \mathcal{O}(k-d))$. We have*

$$s = \dim(f_1U + f_2U) - \binom{k+n-d}{n}.$$

Corollary. *Let T be a line spanned by the polynomials f_1, f_2, and let $k < 2d$. The bundle $V_k|_T$ has splitting type different than $(1, \ldots, 1, 0, \ldots, 0)$ if and only if* $\deg(\gcd(f_1, f_2)) \geq 2d - k$.

Corollary. *Let $k = d + 1$. No types of V_k other than $(1, \ldots, 1, 0, \ldots, 0)$ and $(2, 1, \ldots, 1, 0, \ldots, 0)$ occur. The latter type occurs for lines spanned by polynomials sharing a common factor of degree $d - 1$.*

4. The Set of Jumping Lines

Let $k = d+1$ and $Z \subset \mathbb{Gr}(1, |\mathcal{O}(d)|)$ be the set of lines T such that $V_{d+1}|_T$ has type $(2, 1, \ldots, 1, 0, \ldots, 0)$. We have

$$Z = \operatorname{im}(\mathbb{Gr}(1, |\mathcal{O}(1)|) \times |\mathcal{O}(d-1)| \to \mathbb{Gr}(1, |\mathcal{O}(d)|)),$$

a closed subvariety. For $n \leq 3$, we can calculate its cohomology class:

Theorem. *Let $n \leq 3$ and let $[Z]$ be the class of Z in the Chow ring A. Let b range over the integers with the property $0 \leq b < \frac{\dim Z}{2}$ and define $a = \dim Z - b$, $a' = a + \lfloor \frac{\operatorname{codim} Z}{2} \rfloor$, $b' = b + \lfloor \frac{\operatorname{codim} Z}{2} \rfloor$. Let $\sigma_{a,b}$ be the generating classes of A as in [EH16].*

1. *If $\dim Z$ is odd or $n = 2$, we have*

$$[Z] = \sum_{a,b} \left(\binom{a+1}{n}\binom{b+1}{n} - \binom{a+2}{n}\binom{b}{n} \right) \sigma_{a',b'}. \tag{2}$$

2. *If $\dim Z$ is even and $n = 3$, we have*

$$[Z] = \sum_{a,b} \left(\binom{a+1}{n}\binom{b+1}{n} - \binom{a+2}{n}\binom{b}{n} \right) \sigma_{a',b'} + \binom{\frac{\dim Z}{2}+2}{n}\binom{\frac{\dim Z}{2}}{n} \sigma_{\frac{\dim Z}{2}, \frac{\dim Z}{2}}.$$

References

[EH16] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[Hem15] Christian Hemminghaus. *Families of polarized K3 surfaces and associated bundles*. Master thesis. 2015.